What is the $i\varepsilon$ for the S-matrix?

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based on **hep-th/2204.tomorrow** with Hofie Sigríðar Hannesdóttir

There's a long history in understanding imprints of *causality* on scattering amplitudes

(microcausality, macrocausality, Bogoliubov causality, no Shapiro time advances, …)

[Bogoliubov, Schutzer, Tiomno, van Kampen, Gell-Mann, Goldberger, Thirring, Wanders, Iagolnitzer, Eden, Landshoff, Peres, Branson, Omnes, Chandler, Pham, Stapp, Adams, Arkani-Hamed, Dubovsky, Grinstein, O'Connell, Wise, Giddings, Porto, Camanho, Edelstein, Maldacena, Zhiboedov, Tomboulis, Minwalla, …]

Although never made precise, it is generally believed that causality is encoded in *complex-analytic* properties of the S-matrix

So natural, we no longer consider complexification strange

Multiple practical reasons:

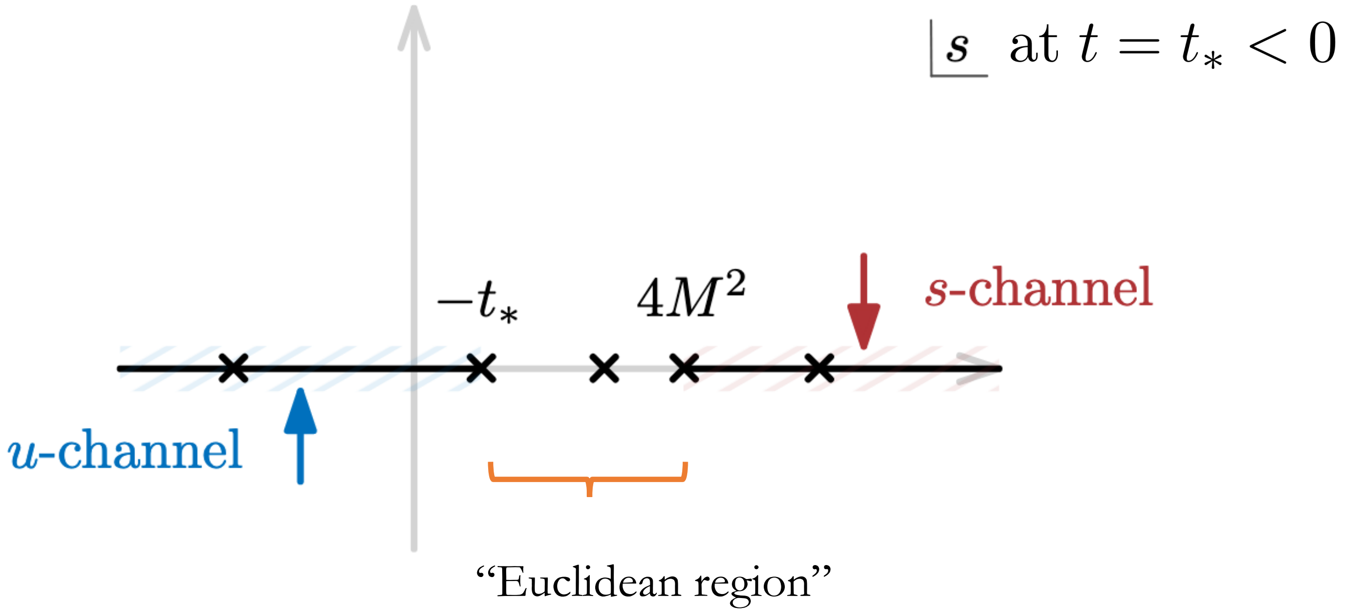
- Theory of complex angular momenta, dispersion relations, on-shell recursion relations, …

- Crossing symmetry

$$\underbrace{e^+e^- \to \gamma\gamma}_{s>0} \qquad \underbrace{\gamma e^- \to \gamma e^-}_{s<0}$$

can we get it "for free"?

Analyticity is best understood for $2 \to 2$ scattering of the lightest state in theories with a mass gap M for low momentum transfer:



$$\mathbf{S}(s, t_*) = \lim_{\varepsilon \to 0^+} \mathbf{S}_{\mathbb{C}}(s + i\varepsilon, t_*)$$

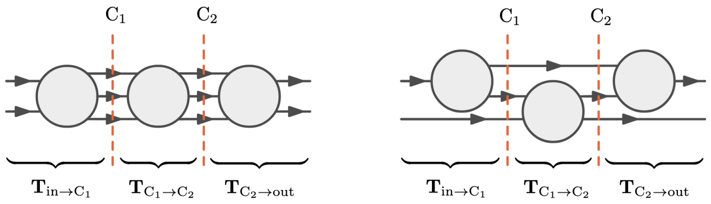
How does this picture extend to more realistic processes?

Standard Model $\supset$ massless and unstable particles, UV/IR divergences, higher-point processes

Outline

- Unitarity constraints

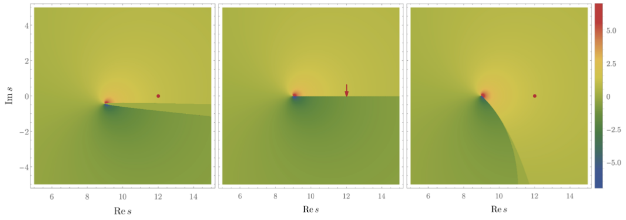
Holomorphic cutting rules



Discontinuities beyond normal thresholds

- Causality constraints

Different ways of implementing causality



Deforming branch cuts in the kinematic space

Notation:

- S-matrix operator: $S = \mathbb{1} + iT$

matrix elements $\mathbf{T} = \langle \text{out}|T|\text{in}\rangle$

- For $2 \to 2$ scattering

$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2 \qquad u = (p_1 + p_3)^2$$

- Subject to momentum conservation

$$s + t + u = \sum_{i=1}^{4} M_i^2$$

Unitarity, $SS^\dagger = \mathbb{1}$, implies that

Sum-integral over all the intermediate states

$$\frac{1}{2i}(T - T^\dagger) = \frac{1}{2}TT^\dagger$$

The RHS is non-holomorphic and doesn't manifest all singularities

Eliminate $T^\dagger = T(\mathbb{1} + iT)^{-1}$ and expand the geometric series

This results in *holomorphic cutting rules*

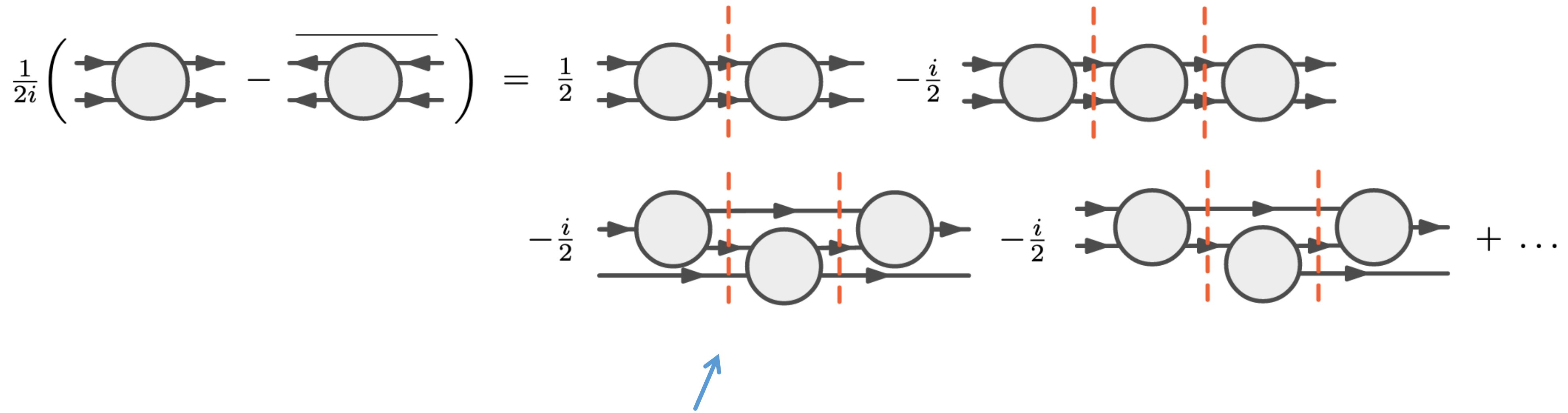
$$\frac{1}{2i}(T - T^\dagger) = -\frac{1}{2}\sum_{c=1}^{\infty}(-iT)^{c+1}$$

number of unitarity cuts

- The place where a new term on the RHS starts contributing is called a *threshold:* a potentially violent event that could give rise to *singularities* or *branch cuts*
- The phase-space is so small, it only allows for *classical* scattering configurations

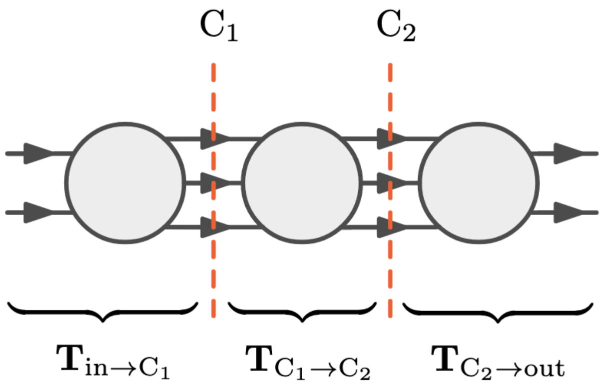
[Coleman, Norton]

Diagrammatically



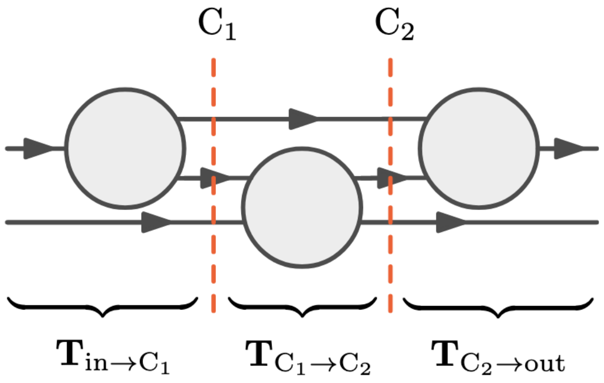
Putting propagators on shell: $\dfrac{-1}{q^2 - m^2 + i\varepsilon} \to 2\pi\delta^+(q^2 - m^2)$

There are two types of thresholds on the RHS:



Normal thresholds

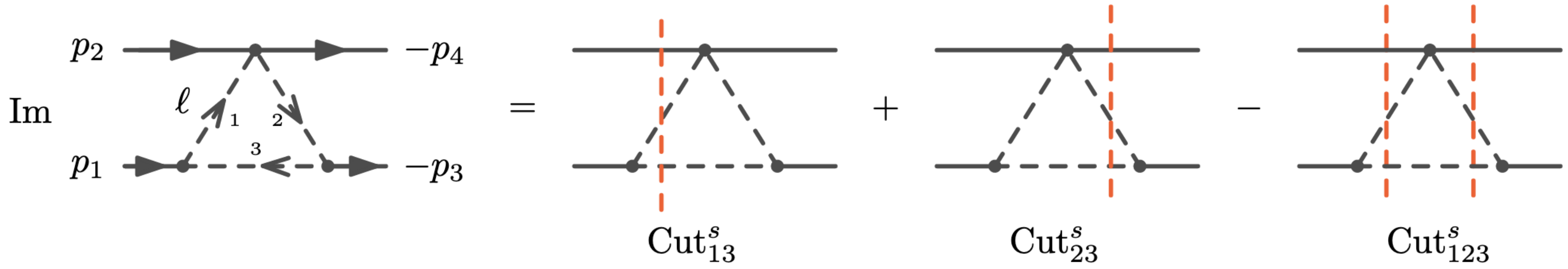
(purely temporal)



"Anomalous" thresholds
or Landau singularities

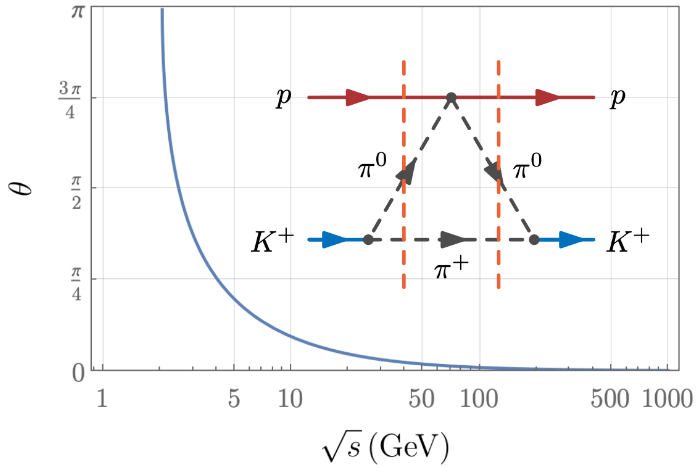
(spatially spread out)

Simplest example:

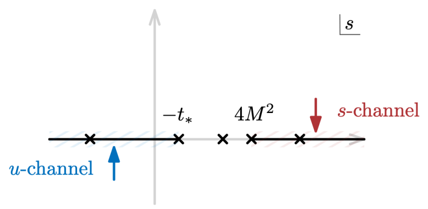


When can we build a triangle diagram with 3 momenta?

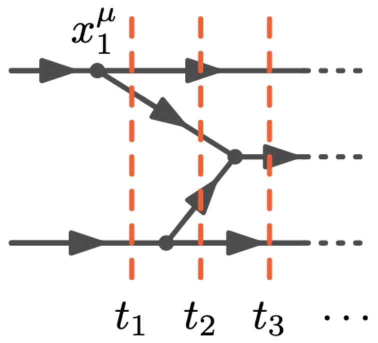
$$\cos\theta = 1 - \frac{2s\left(m_{K^+}^2 - (m_{\pi^0}+m_{\pi^+})^2\right)\left(m_{K^+}^2 - (m_{\pi^0}-m_{\pi^+})^2\right)}{m_{\pi^+}^2\left(s-(m_{K^+}+m_p)^2\right)\left(s-(m_{K^+}-m_p)^2\right)}$$



- Widths move the peak to a complex plane: Breit-Wigner-like distribution
- Heavily suppressed compared to tree-level processes

How is this consistent with  ?

At a threshold, we can time order the interaction vertices:



But if all external particles are stable, we must have *at least*
2 incoming particles interacting at the same vertex:
for $2 \to 2$ this implies only normal thresholds for physical kinematics

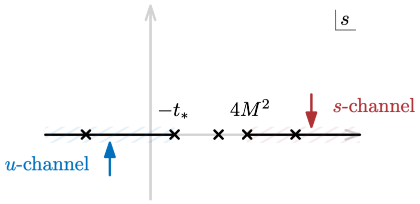
$$s = (m_1 + m_2 + \ldots)^2$$

We need to worry about anomalous thresholds for

- Higher-point scattering
- $2 \to 2$ processes with unstable particles
- Discontinuities of amplitudes
- Branch cuts in analytic expressions

Recent pheno-oriented work includes hadron spectroscopy, $b\bar{b}H$ production, $ZZ \to ZZ$ scattering, …

[Liu, Oka, Zhao, Meissner, Guo, Denner, Dittmaier, Hahn, Boudjema, Ninh, Passarino, …]

It is reasonable to ask how much of the  intuition survives

In particular:

- Can we always uplift the S-matrix to a complex-analytic function in a way consistent with causality?

$$\mathbf{T}(s, t_*) \overset{?}{=} \lim_{\varepsilon \to 0^+} \mathbf{T}_{\mathbb{C}}(s + i\varepsilon, t_*)$$

- Is the imaginary (absorptive) part

$$\mathrm{Im}\,\mathbf{T}(s,t_*) = \tfrac{1}{2i}\Big(\mathbf{T}(s,t_*) - \overline{\mathbf{T}(s,t_*)}\Big)$$

always equal to the discontinuity

$$\mathrm{Disc}_s\,\mathbf{T}_{\mathbb{C}}(s,t_*) = \lim_{\varepsilon\to 0^+} \tfrac{1}{2i}\Big(\mathbf{T}_{\mathbb{C}}(s+i\varepsilon,t_*) - \mathbf{T}_{\mathbb{C}}(s-i\varepsilon,t_*)\Big) \quad ?$$

Where do we even start?

Convert into algebraic problems for every Feynman diagram:

We'll explain these conditions on the next slides

$$
\begin{cases}
\mathcal{V} = 0 & \text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch cut} \\
\partial_{\alpha_e} \mathcal{V} = 0 & \text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch point} \\
\mathrm{Im}\, \mathcal{V} > 0 & \text{for all } \alpha\text{'s} \quad \Leftrightarrow \quad \text{causal branch}
\end{cases}
$$

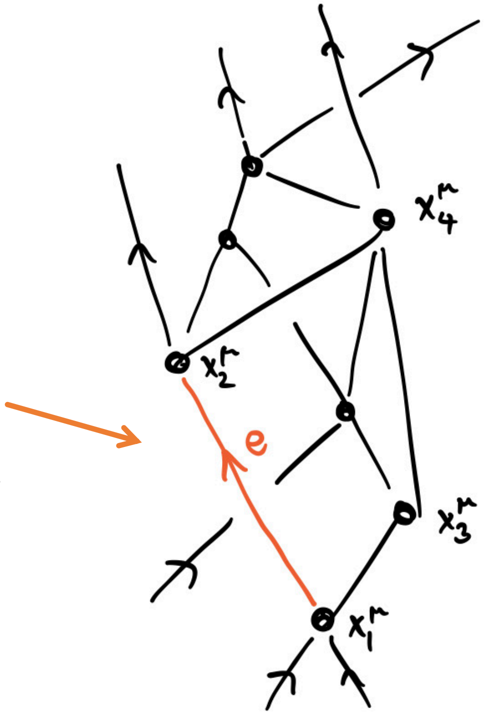
We already know *branch points* are classical scattering configurations:

Momentum q_e^μ

Mass m_e

Schwinger proper time $\alpha_e \geqslant 0$

Space-time displacement $\Delta x_e^\mu = \alpha_e q_e^\mu$



Momentum conservation at every vertex:

$$\sum_{e \ni v} q_e^\mu + \sum_{i \ni v} p_i^\mu = 0$$

Local interactions at vertices:

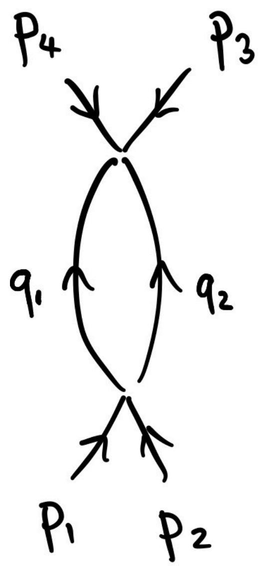
$$x_j^\mu - x_i^\mu = \sum_{e: i \to j} \Delta x_e^\mu$$

On-shell conditions for every edge:

$$q_e^2 - m_e^2 = 0$$

Landau equations [Bjorken, Landau, Nakanishi]

Simplest example:



$$s = (p_1 + p_2)^2$$

momentum conservation

$$p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu = -p_3^\mu - p_4^\mu$$

locality

$$\alpha_1 q_1^\mu = \alpha_2 q_2^\mu$$

on-shellness

$$q_1^2 - m_1^2 = 0$$
$$q_2^2 - m_2^2 = 0$$

$$q_1^\mu = \ell^\mu, \qquad q_2^\mu = p_1^\mu + p_2^\mu - \ell^\mu$$

$$\ell^\mu = (p_1^\mu + p_2^\mu)\frac{\alpha_2}{\alpha_1+\alpha_2}$$

$$s\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2 - m_1^2 = 0, \qquad s\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 - m_2^2 = 0$$

Lorentz invariant

Can be concisely summarized as:

$$\begin{cases} s\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2 - m_1^2 = 0, \\ s\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 - m_2^2 = 0 \end{cases} \iff \begin{cases} \mathcal{V} = s\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - m_1^2\alpha_1 - m_2^2\alpha_2 \\ \partial_{\alpha_1}\mathcal{V} = 0 \\ \partial_{\alpha_2}\mathcal{V} = 0 \end{cases}$$

The solutions are

$$(\alpha_1 : \alpha_2) = \left(\frac{1}{m_1} : \pm\frac{1}{m_2}\right) \qquad s = (m_1 \pm m_2)^2$$

Note projective invariance in Schwinger parameters and kinematic variables separately

$+$ normal threshold

$-$ pseudo-normal threshold

In practice, Schwinger parametrization of the bubble integral gives:

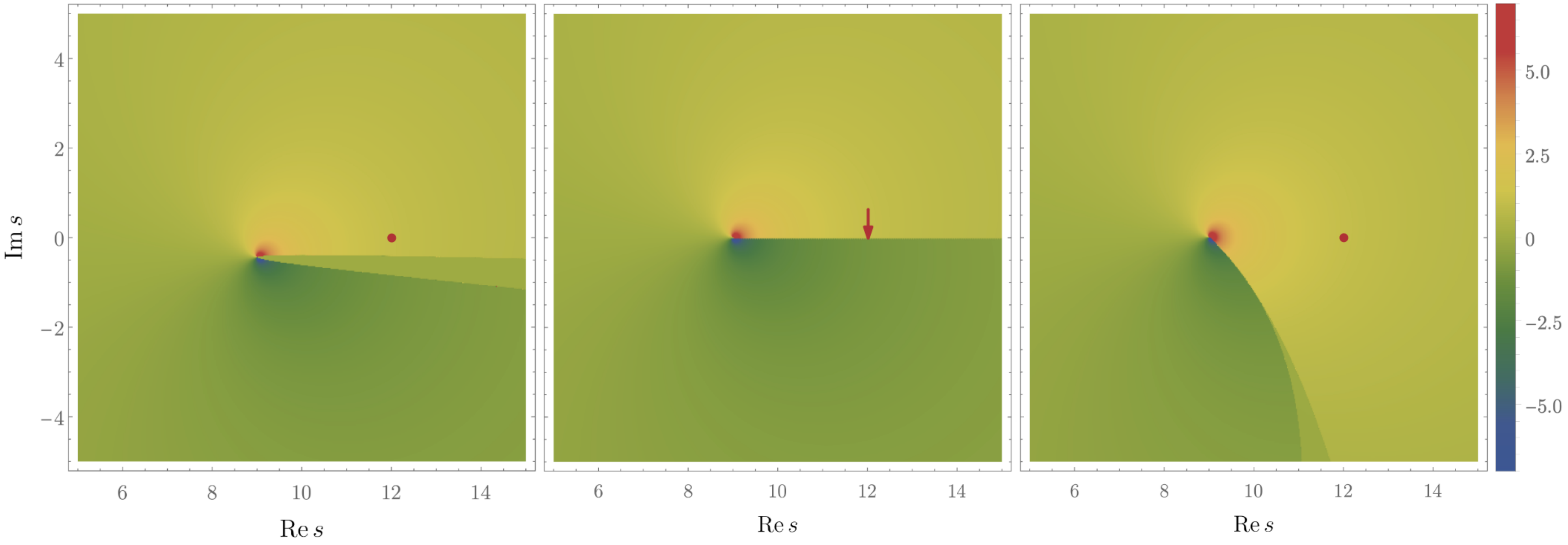
$$\int_0^\infty \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{\mathcal{V}^{2-\mathrm{D}/2}}\delta(\alpha_1+\alpha_2-1) \qquad \text{with} \qquad \mathcal{V} = s\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - m_1^2\alpha_1 - m_2^2\alpha_2$$

- When $\mathcal{V}=0$, we have to make a decision how to deform away from it (branch cut)

- Causal branch determined by $\mathrm{Im}\,\mathcal{V}>0$

There are three options for implementing $\operatorname{Im}\mathcal{V} > 0$ $\left(\mathcal{V} = s\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - m_1^2\alpha_1 - m_2^2\alpha_2\right)$:

$m_e^2 \to m_e^2 - i\varepsilon$ | $s \to s + i\varepsilon$ | $\alpha_e \to \hat{\alpha}_e$



Feynman $i\varepsilon$
(moves branch points, unphysical)

Kinematic $s{+}i\varepsilon$
(doesn't work in general)

Branch cut deformations

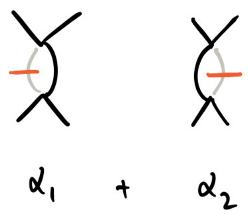
This structure is not a coincidence!
For any Feynman diagram we can define the *worldline action*

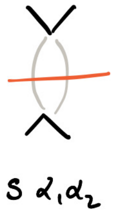
$$\mathcal{V}(\alpha_e; s_{ij}, m_e) = \frac{\mathcal{F}}{\mathcal{U}},$$

where the two *Symanzik polynomials* are given by

for the experts

$$\mathcal{U} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} \alpha_e, \qquad \mathcal{F} = \sum_{\substack{\text{spanning} \\ \text{2-trees } T_L \sqcup T_R}} p_L^2 \prod_{e \notin T_L, T_R} \alpha_e - \mathcal{U} \sum_{e=1}^{\mathrm{E}} m_e^2 \, \alpha_e$$





$$\Rightarrow \; \mathcal{V} = s \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} - m_1^2 \alpha_1 - m_2^2 \alpha_2$$

Summary so far:

Worldline action

Extremizing gives a classical saddle point

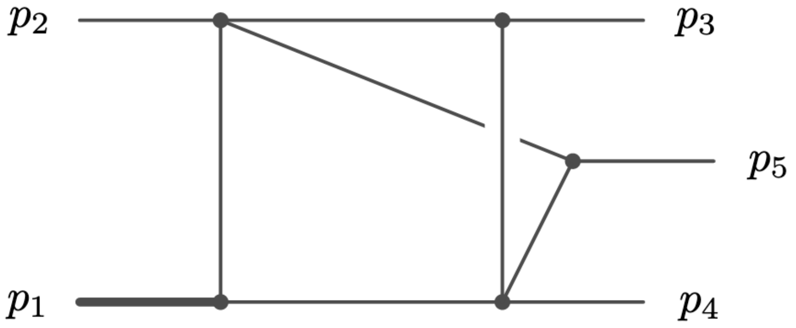
How to implement consistently?

$$\begin{aligned} \mathcal{V} &= 0 \quad && \text{for any } \alpha\text{'s} \quad && \Leftrightarrow \quad && \text{branch cut} \\ \partial_{\alpha_e} \mathcal{V} &= 0 \quad && \text{for any } \alpha\text{'s} \quad && \Leftrightarrow \quad && \text{branch point} \\ \operatorname{Im} \mathcal{V} &> 0 \quad && \text{for all } \alpha\text{'s} \quad && \Leftrightarrow \quad && \text{causal branch} \end{aligned}$$

Analytic properties can be studied without explicit computations

Nowadays we have powerful algebraic geometry tools to address such questions

[SM, Telen ‘21]



$$\Sigma_5 = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$

Two-Loop Hexa-Box Integrals for Non-Planar Five-Point One-Mass Processes

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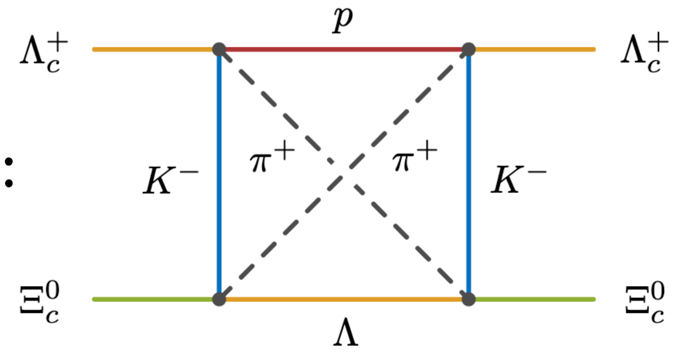
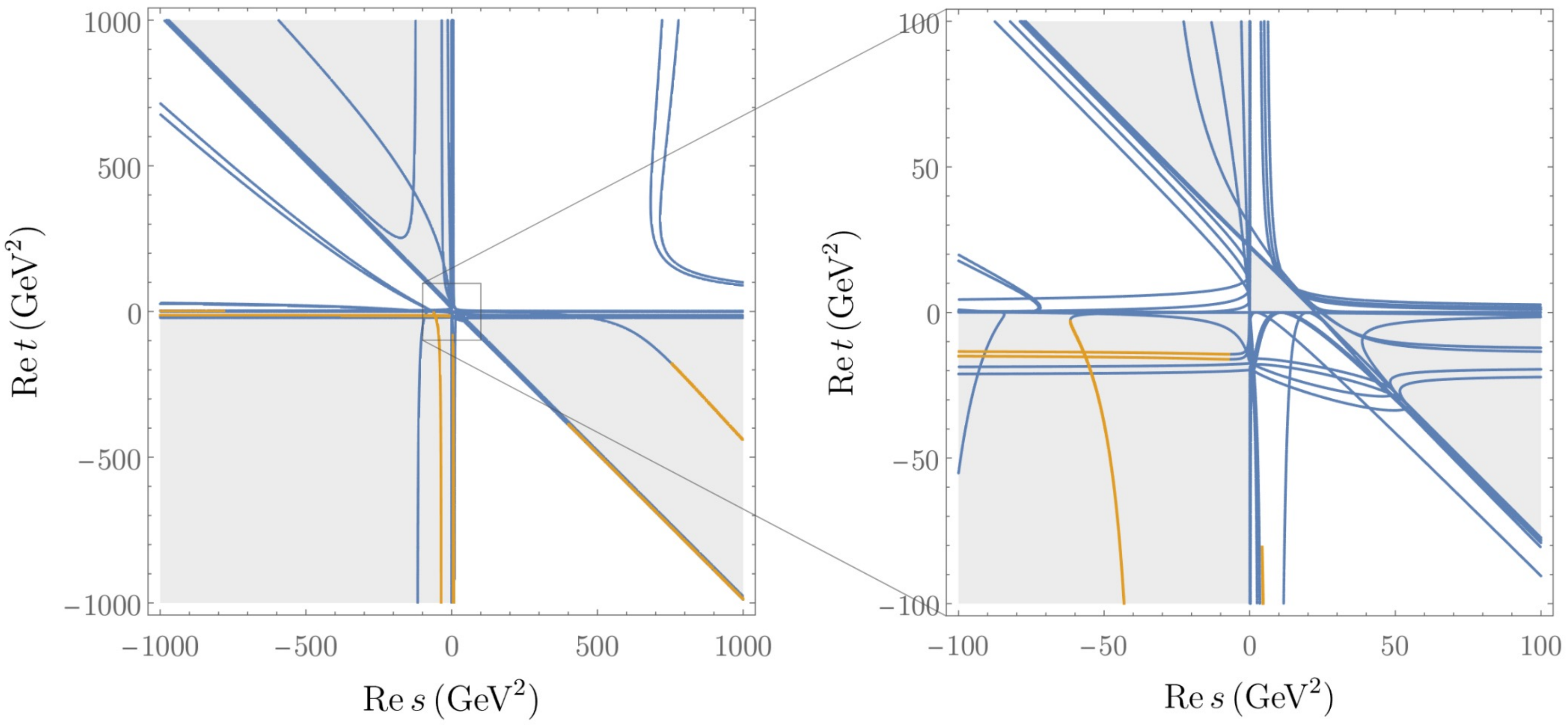
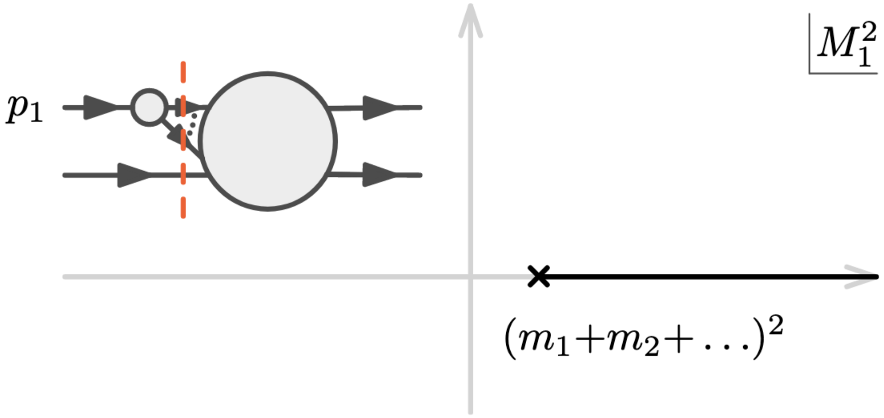
[hep-ph/2107.14180]

They very quickly get out of hand:





Why couldn't we just use $s + i\varepsilon$? First sign of problems:



Off-shell:
branch cut between

$$M_1^2 \pm i\varepsilon$$

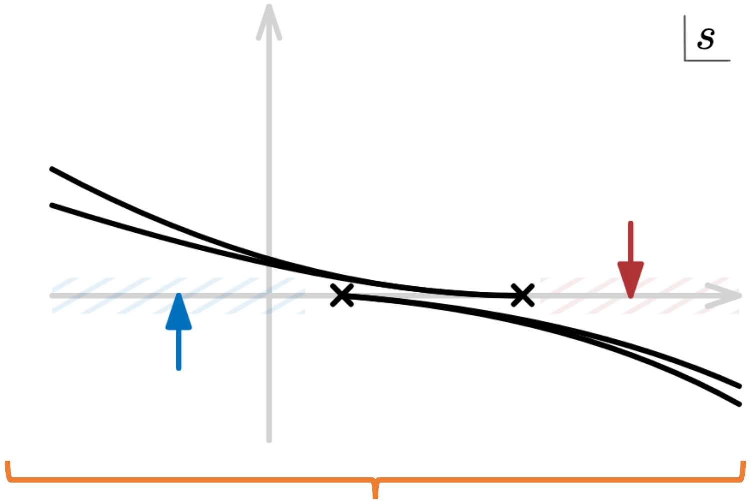
By momentum conservation

$$(s \mp i\varepsilon) + t + u = \sum_{i=1}^{4} M_i^2$$

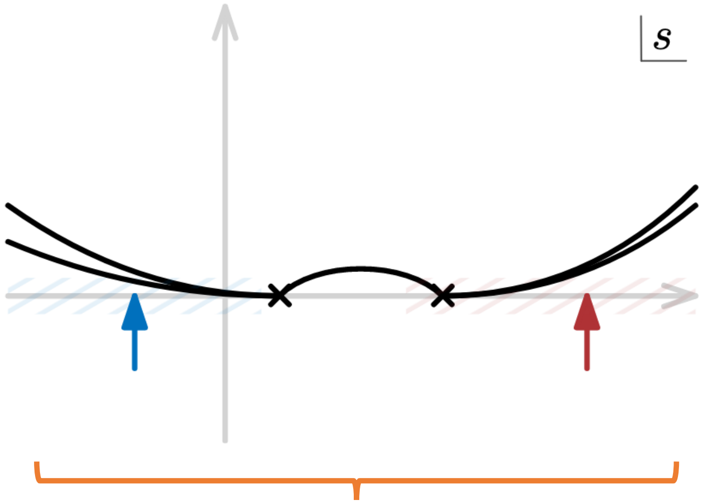
On-shell:
branch cut between

$$s \mp i\varepsilon$$

Once we encounter a branch cuts for all s, there are two possibilities:



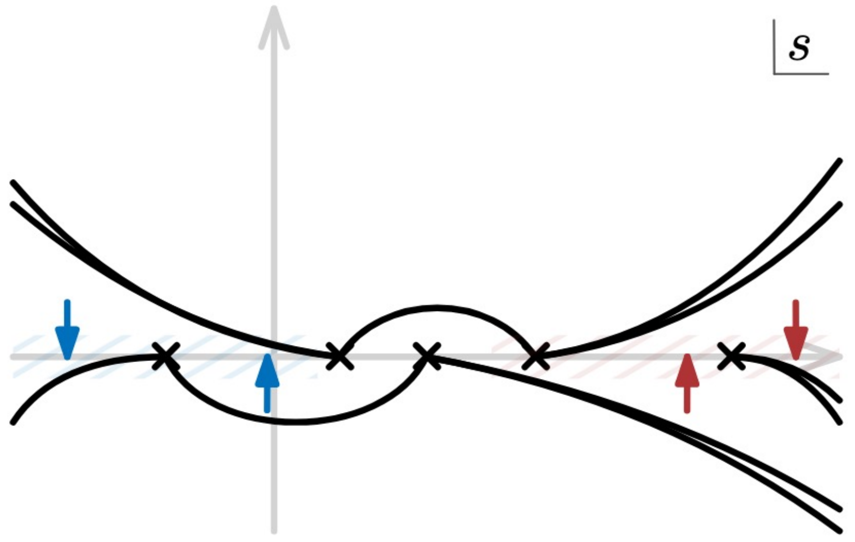
Can connect upper- and lower-half planes



Cannot connect
(two distinct analytic functions)

There's no unique way to approach physical regions!

We are forced to perform branch cut deformations:



physical sheet

Causality: giving worldlines a small phase [SM '21]

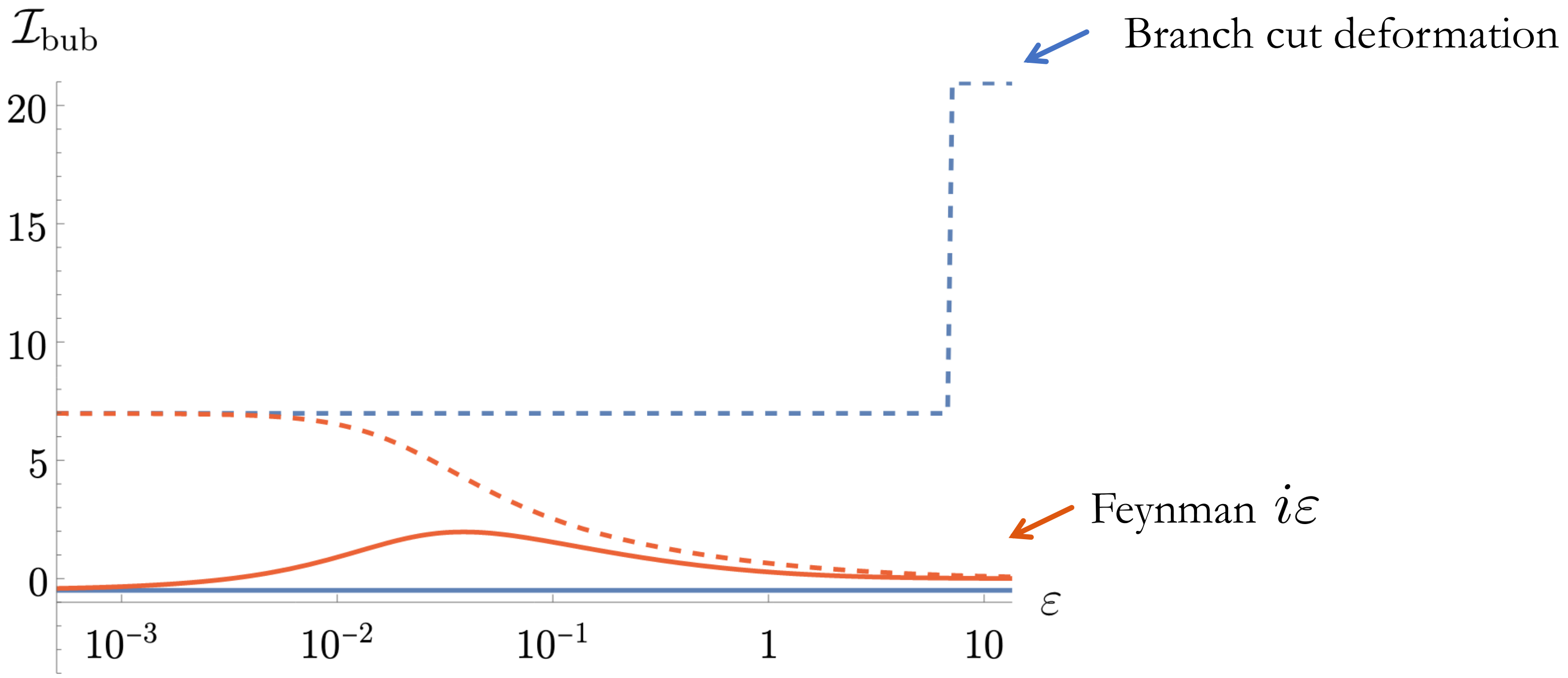
$$\alpha_e \to \alpha_e \exp(i\varepsilon \partial_{\alpha_e} \mathcal{V})$$
$$= \alpha_e \left(1 + i\varepsilon \partial_{\alpha_e} \mathcal{V} + \ldots\right)$$

At the level of the action:

$$\mathcal{V} \to \mathcal{V} + i\varepsilon \underbrace{\sum_{e=1}^{\mathrm{E}} \alpha_e (\partial_{\alpha_e} \mathcal{V})^2}_{\geqslant 0} + \ldots$$

← Breaks down directly at the branch points

In practice, we only need a *sufficiently small* ε (as opposed to infinitesimal)



Using this technique, one can show two general results:

- $2 \to 2$ scattering with no unstable external particles:

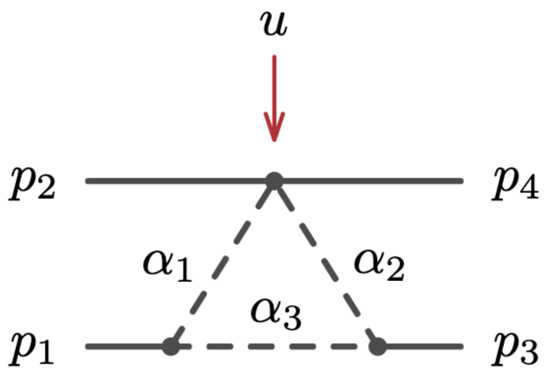
$$\mathbf{T}(s,t_*) = \lim_{\varepsilon \to 0^+} \mathbf{T}_{\mathbb{C}}(s+i\varepsilon, t_*) \qquad \mathrm{Im}\,\mathbf{T} = \mathrm{Disc}_s \mathbf{T}_{\mathbb{C}}$$

(previously only established when the Euclidean region exists)

- $2 \to 2$ scattering with unstable external particles:

$$\mathbf{T}(s,t_*) \neq \lim_{\varepsilon \to 0^+} \mathbf{T}_{\mathbb{C}}(s \pm i\varepsilon, t_*) \qquad \mathrm{Im}\,\mathbf{T} \neq \mathrm{Disc}_s \mathbf{T}_{\mathbb{C}}$$

The simplest example:

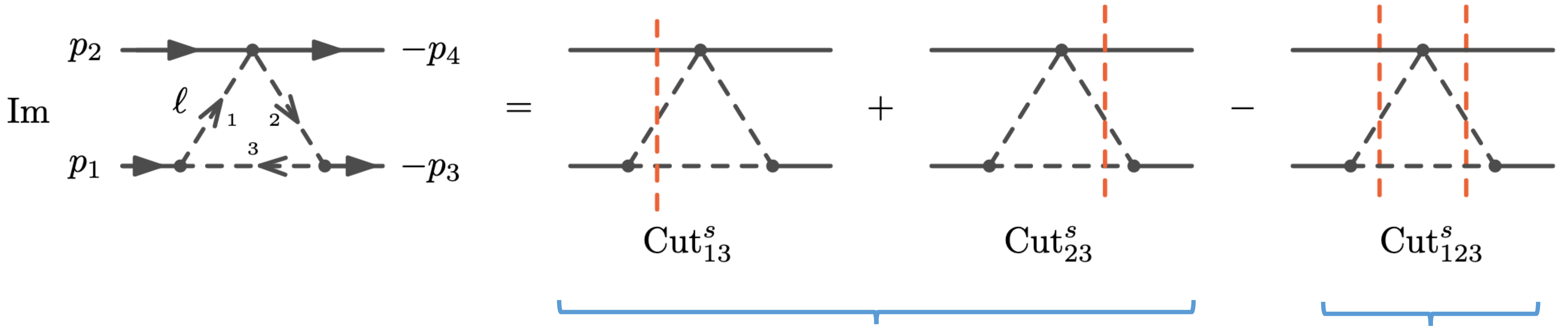


Internal masses m

External masses $M > 2m$

Action: $$\mathcal{V} = \frac{u\alpha_1\alpha_2 + M^2\alpha_3(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2+\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3)$$

Unitarity in the s-channel:



External mass singularity
(present for any s)

Triangle threshold

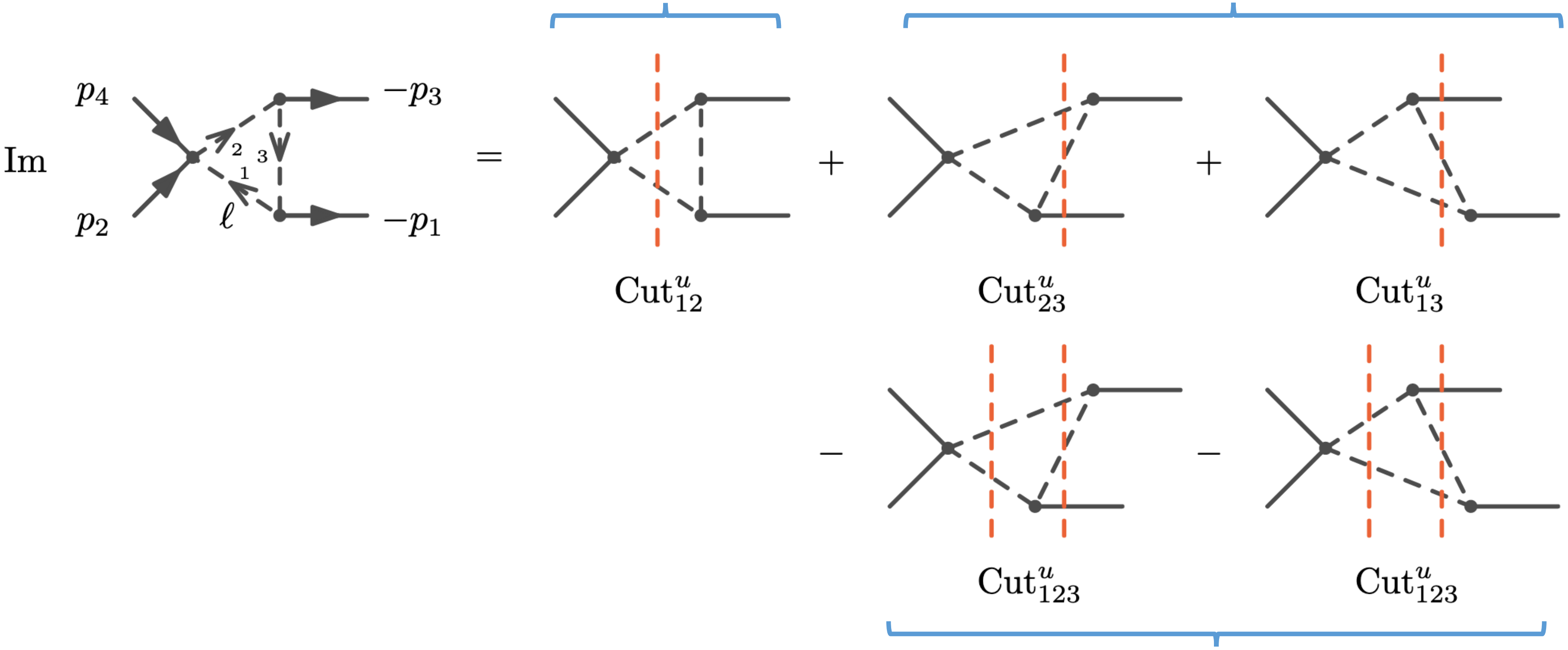
$$s = \frac{M^4}{m^2} - t_*$$

Unitarity in the u-channel:

Normal threshold

$s = 4(M^2 - m^2) - t_*$

External mass singularity
(present for any s)



Triangle threshold (forbidden kinematically)

Two distinct analytic functions in the UHP and LHP:

$$\mathcal{I}_{\rm tri}^{\rm UHP}(s,t) = \frac{z}{4M^2\beta_z}\sum_{\zeta\in\{-1,1\}}\Bigg\{\zeta\,\mathrm{Li}_2\left(\frac{1+\frac{z}{2}-\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)+\zeta\,\mathrm{Li}_2\left(1-\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)$$
$$+2\,\mathrm{Li}_2\left(\frac{1+\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\,\mathrm{Li}_2\left(\frac{1-\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)+2\pi i\log\left(\frac{1+\beta_z}{1+\beta_z\beta_{yz}}\right)$$
$$+\zeta\log\left(\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\left[-\pi i+\log\left(-1+\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\right]\Bigg\},$$

$$\mathcal{I}_{\rm tri}^{\rm LHP}(s,t) = \frac{z}{4M^2\beta_z}\sum_{\zeta\in\{-1,1\}}\Bigg\{\zeta\,\mathrm{Li}_2\left(\frac{1+\frac{z}{2}-\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)+\zeta\,\mathrm{Li}_2\left(1-\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)$$
$$+2\,\mathrm{Li}_2\left(\frac{1+\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\,\mathrm{Li}_2\left(\frac{1-\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\pi i\log\left(\frac{1+\beta_z}{1+\beta_z\beta_{yz}}\right)$$
$$+\zeta\log\left(\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\left[\pi i+\log\left(-1+\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\right]\Bigg\}.$$

where

$$y=-\frac{4m^2}{u}\,,\qquad z=-\frac{4M^2}{u}\,,$$
$$\beta_y=\sqrt{1+y}\,,\qquad \beta_z=\sqrt{1+z}\,,\qquad \beta_{yz}=-i\sqrt{-1+\frac{4y}{z}}\,.$$

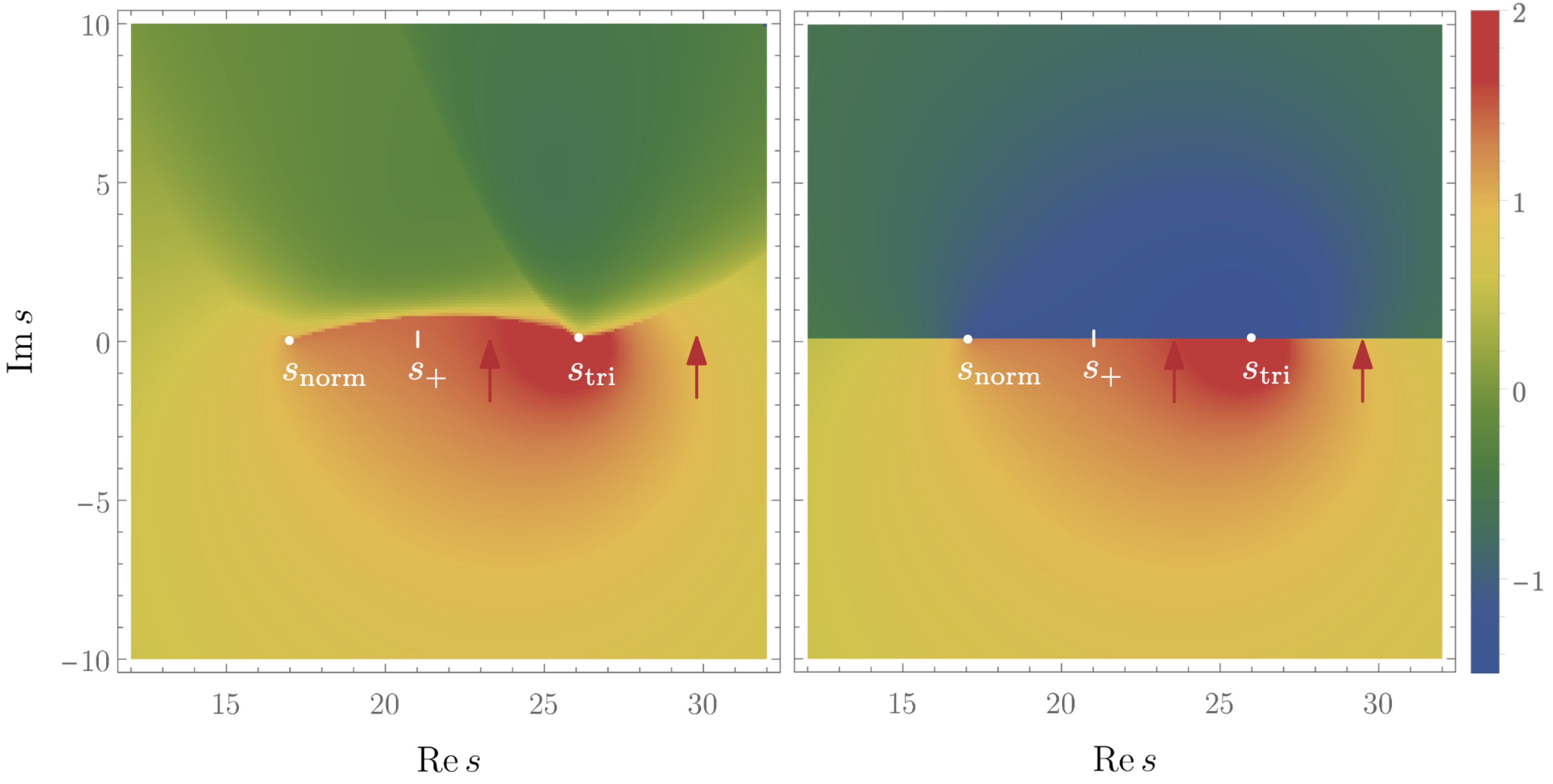
Causality requires $\mathrm{Im}\mathcal{V} > 0$ ($\mathcal{V} = \frac{u\alpha_1\alpha_2 + M^2\alpha_3(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2+\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3)$):

$$\mathrm{Im}\,\mathcal{V} = \mathrm{Im}\,u \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2+\alpha_3}$$

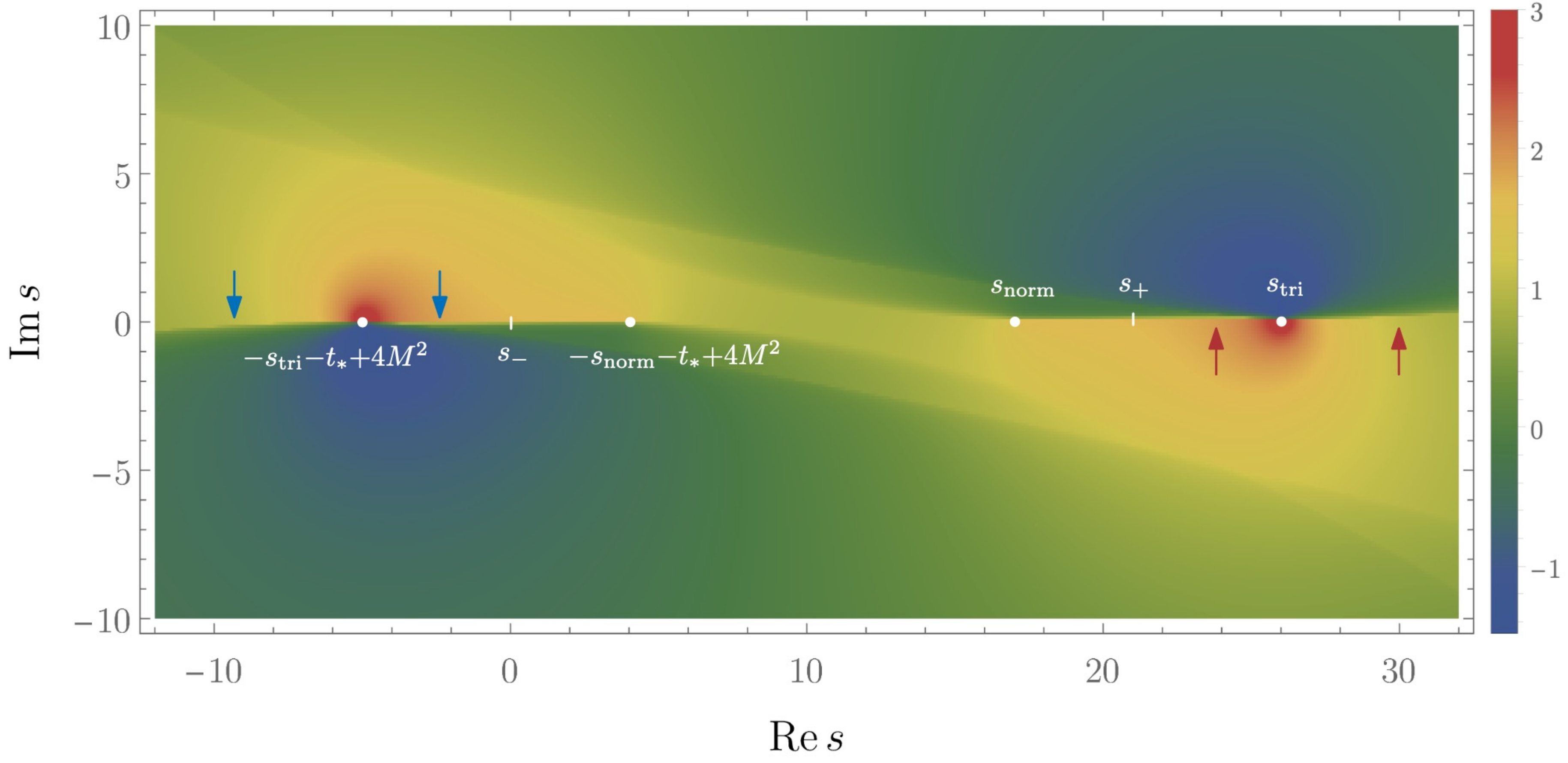
$$= -\underbrace{\mathrm{Im}\,s}_{<0} \underbrace{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2+\alpha_3}}_{>0} > 0$$

Approach both s- and u-channel physical regions from LHP

Comparing numerical and analytic expressions:



Finally, summing over multiple Feynman diagrams



Many open questions, for example:

- How big of a mistake we'd make by always approaching the s-channel from the UHP?

$$\propto \left(\frac{\Gamma}{M}\right)^{\#}$$

- Effect on practical Standard Model computations?

e.g., $ZZ \to ZZ$

- What is the analogue for $2 \to 3$ scattering?

[If there's time]

Since singularities are already determined by saddle points

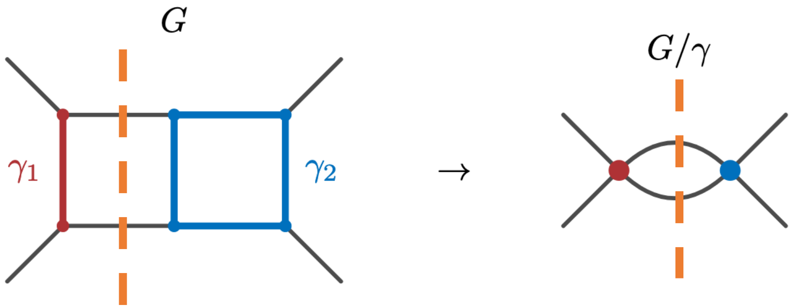
$$\alpha_e = \alpha_e^*, \qquad \Delta(s, t, M, m) = 0$$

Why don't we just study fluctuations around such saddles:

$$\alpha_e = \alpha_e^* + \delta\alpha_e + \ldots, \qquad \Delta(s, t, M, m) = 0 + \delta\Delta + \ldots$$

Local behavior around the threshold

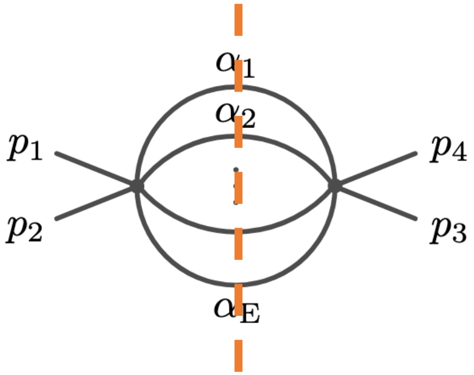
So far limited to isolated and non-degenerate saddles
(excludes massless Feynman integrals)



$$\mathcal{I}_G \approx \# \prod_i \mathcal{I}^*_{\gamma_i} \begin{cases} \Delta^\rho & \text{if } \rho < 0 \\ \log \Delta & \text{if } \rho = 0 \end{cases}$$

where $\rho = \dfrac{\mathrm{L}_{G/\gamma}\mathrm{D} - \mathrm{E}_{G/\gamma} - 2d_{\mathcal{N}_{G/\gamma}} - 1}{2}$

For example, near every normal threshold



$$\mathcal{I}_G \approx \# \mathcal{I}^*_{\gamma_L} \mathcal{I}^*_{\gamma_R} \begin{cases} \Gamma(-\rho) \left[\left(\sum_{e=1}^{E} m_e \right)^2 - s \right]^{\rho} & \text{if } D < \frac{E+1}{E-1}, \\ -\log \left[\left(\sum_{e=1}^{E} m_e \right)^2 - s \right] & \text{if } D = \frac{E+1}{E-1}, \end{cases}$$

$$\text{where} \quad \rho = \frac{(E-1)D - E - 1}{2}$$

Naively, Δ^ρ would suggest that the S-matrix can have arbitrarily-singular behavior…

We're rescued if we assume analyticity (at most codim-1 singularities): $\mathrm{E}_{G/\gamma} - \mathrm{L}_{G/\gamma}\mathrm{D} \leqslant 1$

$$\rho = \frac{\mathrm{L}_{G/\gamma}\mathrm{D} - \mathrm{E}_{G/\gamma} - 2d_{\mathcal{N}_{G/\gamma}} - 1}{2} \geqslant -1$$

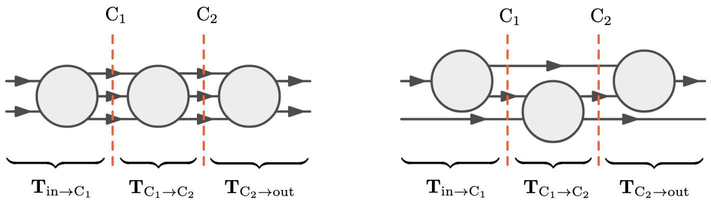
Every 1VI component can only lead to singularities of the type

$$\frac{1}{\Delta}, \qquad \frac{1}{\sqrt{\Delta}}, \qquad \log \Delta$$

Summary

- Unitarity constraints

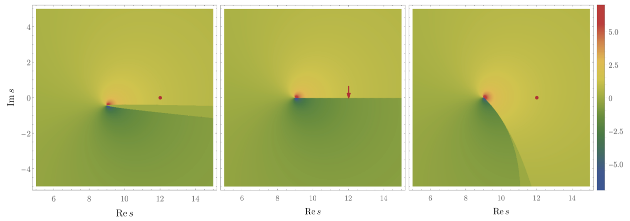
Holomorphic cutting rules



Discontinuities beyond normal thresholds

- Causality constraints

Different ways of implementing causality



Deforming branch cuts in the kinematic space

Thank you