Tropical Realizations of the Wilsonian Paradigm

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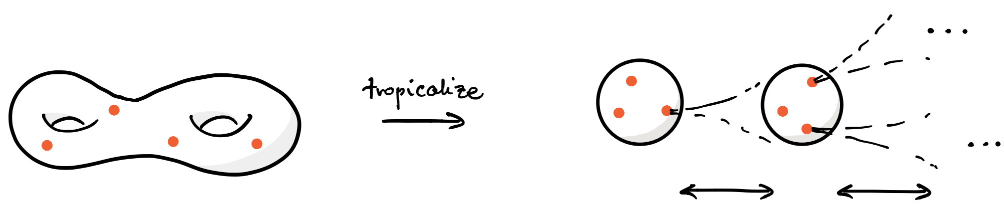
with Nima Arkani-Hamed and Aaron Hillman

Appearance of tropical geometry in scattering amplitudes

1. Worldsheets:

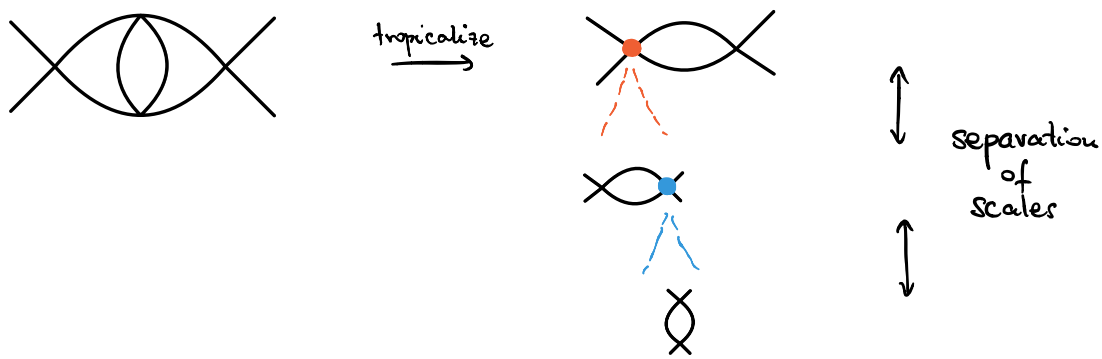


OPE-like factorization

[Tourkine '13]

[other configuration spaces Cachazo, Early, Guevara,
Arkani-Hamed, He, Lam, Thomas, Spradlin, ... '19-]

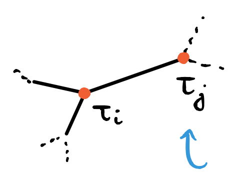
2. Worldlines



[related work Schultka, Panzer, Borinsky, ... '18–]

Stark contrast with the worldsheet picture:

dim-less (Weyl invariance)



proper time

$$G_{ij} \sim \log|z_i - z_j| + \dots$$

$$G_{ij} \sim |\tau_i - \tau_j| + \dots$$

mass scale!

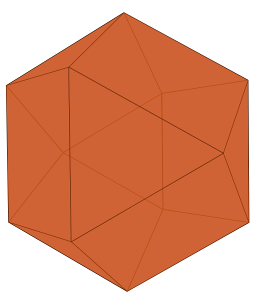
Central idea:

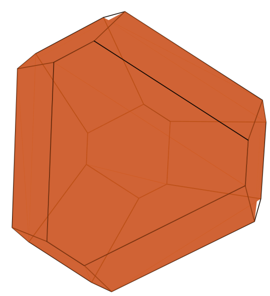
Tropical geometry is a realization of the idea of the separation of scales

Multi-faceted story: renormalization, soft-collinear physics, Landau analysis, worldline formalism, ...

UV / IR divergences

threshold singularities



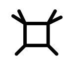


[with Arkani-Hamed & Hillman]

[with Telen]

this talk (simple examples only)

Plan

- → Review of Schwinger parametrization
- → UV divergences 
- → IR divergences 

Feynman integrals in the worldline formalism give:

Schwinger parameters $\alpha_{e=1,2,\dots,E}$

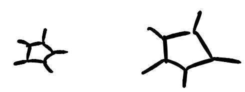
$$\Gamma(\gamma) \int \underbrace{\frac{d^E\alpha}{GL(1)}}_{\text{projective measure on } \mathbb{CP}^{E-1}} \frac{N}{\mathcal{U}^{D/2-\gamma}\,\mathcal{F}^{\gamma}}$$

polynomials in α_e

degree of divergence

$\gamma = E - LD/2 + \deg N$

→ Overall $GL(1)$ redundancy : $\alpha_e \to \lambda\, \alpha_e$



Fix by setting $\alpha_E = 1$, so $\displaystyle\int \frac{d^E\alpha}{GL(1)} = \int_0^\infty d^{E-1}\alpha$

→ Symanzik polynomials

$$\mathcal{U} = \sum_{\substack{\text{spanning}\\ \text{trees } T}} \prod_{e \notin T} \alpha_e$$

$$\mathcal{F} = \sum_{\substack{\text{spanning}\\ \text{2-trees}\\ T_L \cup T_R}} p_L^2 \prod_{e \notin T_L, T_R} \alpha_e - \Big(\sum_e m_e^2\, \alpha_e\Big)\, \mathcal{U}.$$

Such integrals don't converge for two reasons:

→ Threshold singularities. Needs the "Feynman $i\varepsilon$"

$$\int_0^{\infty} d\alpha_e \quad \longrightarrow \quad \int_0^{\infty \exp(i\varepsilon\, \partial F/\partial\alpha_e)} d\alpha_e$$

[SM '21]

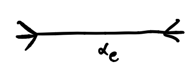
Imposes correct causality conditions unless $\dfrac{\partial F}{\partial \alpha_e} = 0 \quad \forall e$

$\Longleftrightarrow$ Landau equations

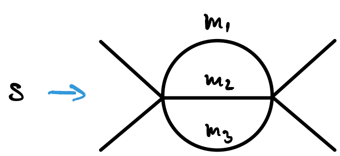
→ UV/IR divergences. In this talk we'll employ dimensional regularization

$$D = 4 - 2\epsilon, \qquad \epsilon \in \mathbb{C} \setminus \mathbb{Z}$$

Schematically: $\alpha_e \to 0$ UV 

$\alpha_e \to \infty$ IR 

UV Example



$E = 3$
$L = 2$
$D = 4-2\epsilon$

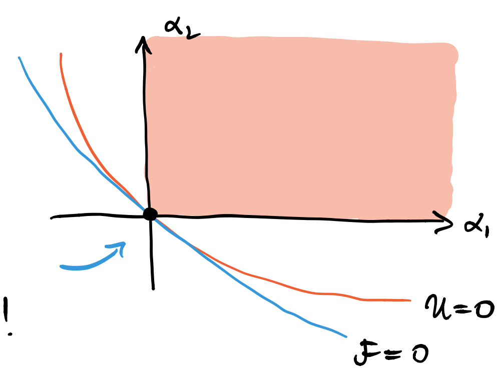
$$\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 ,$$

$$\mathcal{F} = s\,\alpha_1\alpha_2\alpha_3 - \left(m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3\right)\mathcal{U} .$$

The Feynman integral reads

$$\Gamma(-1+2\epsilon) \int_0^\infty \frac{d\alpha_1\, d\alpha_2}{\mathcal{U}^3\, \mathcal{F}^{-1}} \left(\frac{\mathcal{U}^3}{\mathcal{F}^2}\right)^{\epsilon} \Bigg|_{\alpha_3=1}$$

Looking for divergences naively:



the devil's
in the detail!

Standard approach: iterated blow-up of all boundaries

[sector decomposition: Binoth, Heinrich, Bogner, Weinzierl, Kaneko, Ueda ...]

[diagramatic version by Bloch, Esnault, Kreimer '13 Brown '15]

Tropical compactification gives us the blow-up for free.
Let's "discover" it in a couple of steps.

1. Push all divergences to infinity:

$$\alpha_e = e^{\tau_e}$$

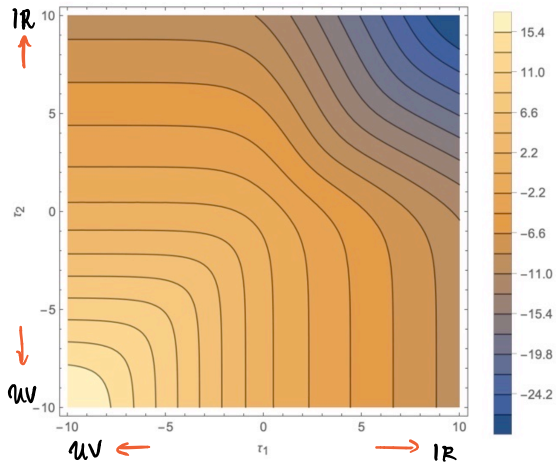
For which we have

$\tau_e \to -\infty$ UV

$\tau_e \to +\infty$ IR

(but recall that $\alpha_e \sim \lambda \alpha_e$ so $\tau_e \sim \tau_e + \log \lambda$)

Let's plot the log of the integrand in (τ_1, τ_2) variables with $\tau_3 = 0$

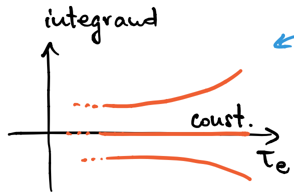


Still not illuminating because it's difficult to read-off the rates of divergence at infinity

2. Factor out the overall log divergence :

$$\int_0^\infty \frac{d\alpha_1\, d\alpha_2}{\alpha_1 \alpha_2 \alpha_3} \underbrace{\frac{\alpha_1 \alpha_2 \alpha_3}{\mathcal{U}^3 \mathcal{F}^{-1}}}_{\text{plot this}} \left(\frac{\mathcal{U}^3}{\mathcal{F}^2} \right)^\epsilon \Bigg|_{\alpha_3 = 1}$$

Expectation:



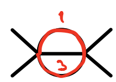
quadratic & higher

log divergence

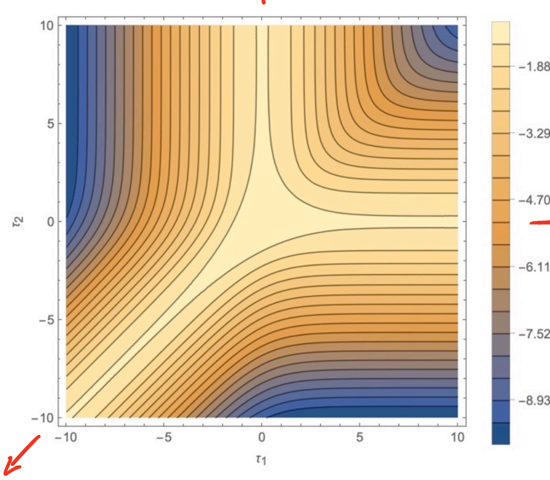
convergent

We find three directions where the normalized integrand goes to a constant:

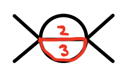
$\tau_2 \to \infty$ or $\tau_1 \to -\infty$, $\tau_3 \to -\infty$



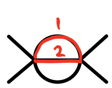
"amoeba"



$\tau_1 \to \infty$ or $\tau_2 \to -\infty$, $\tau_3 \to -\infty$



$\tau_1 \to -\infty$
$\tau_2 \to -\infty$



3. Divergences are associated with an enhanced symmetry in certain directions, e.g., when $\alpha_1 , \alpha_2 \ll \alpha_3 = 1$:

$$\mathcal{U} = \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1$$
$$\to \alpha_3 (\alpha_1 + \alpha_2) + \dots$$

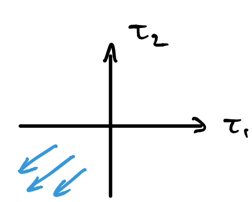
$$\mathcal{F} = s\, \alpha_1 \alpha_2 \alpha_3 - (m_1^2 \alpha_1 + m_2^2 \alpha_2 + m_3^2 \alpha_3)\, \mathcal{U}$$
$$\to - m_3^2\, \alpha_3^2 (\alpha_1 + \alpha_2) + \dots$$

Therefore the integrand can be approximated with

$$\frac{\alpha_1 \alpha_2 \alpha_3}{\mathcal{U}^3 \mathcal{F}^{-1}} \left(\frac{\mathcal{U}^3}{\mathcal{F}^2} \right)^{\epsilon} \to - m_3^2 \underbrace{\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2}}_{\text{additional } GL(1) \text{ symmetry}} \underbrace{\left(\frac{\alpha_1 + \alpha_2}{m_3^4} \right)^{\epsilon}}_{\text{regulator}} + \dots$$

$$\alpha_1 \to \lambda \alpha_1, \quad \alpha_2 \to \lambda \alpha_2 \qquad \text{with } \lambda \ll 1 .$$

$$\left(\text{or } \begin{aligned} \tau_1 &\to \tau_1 + \log \lambda \\ \tau_2 &\to \tau_2 + \log \lambda \end{aligned} \right).$$



Hence these parts of the integration region give

$$\int \frac{d^3\alpha}{GL(1)} \, (\ldots) = \infty \qquad \left(\text{or} \sim \frac{1}{\epsilon} \text{ in dim. reg.}\right)$$

extra GL(1)

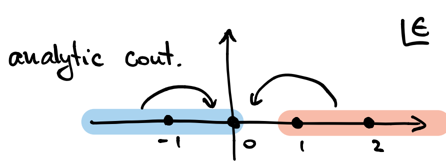
We need to learn how to "mod out" by an additional GL(1)

$$\int \frac{d^3\alpha}{GL(1)^2} \, (\ldots) = \text{coefficient of } 1/\epsilon .$$

But wait!

There's something wrong with the logic ...

Calling $(\ldots)^\epsilon$ a "regulator" assumes that ϵ is infinitesimal, but in general $\epsilon \in \mathbb{C} \setminus \mathbb{Z}$



In other words, approximating the integrand does not approximate the integral, if the latter doesn't converge! It only works for log-divergences ...

4. Since divergences are at infinity, we need to focus on

$$\alpha_e = e^{\delta \tau_e} \quad \text{as} \quad \delta \to \infty$$

e.g. $\mathcal{U} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$

$$= e^{\delta(\tau_1+\tau_2)} + e^{\delta(\tau_2+\tau_3)} + e^{\delta(\tau_3+\tau_1)}$$

$$\to e^{\delta \max(\tau_1+\tau_2,\ \tau_2+\tau_3,\ \tau_3+\tau_1)} + \ldots$$

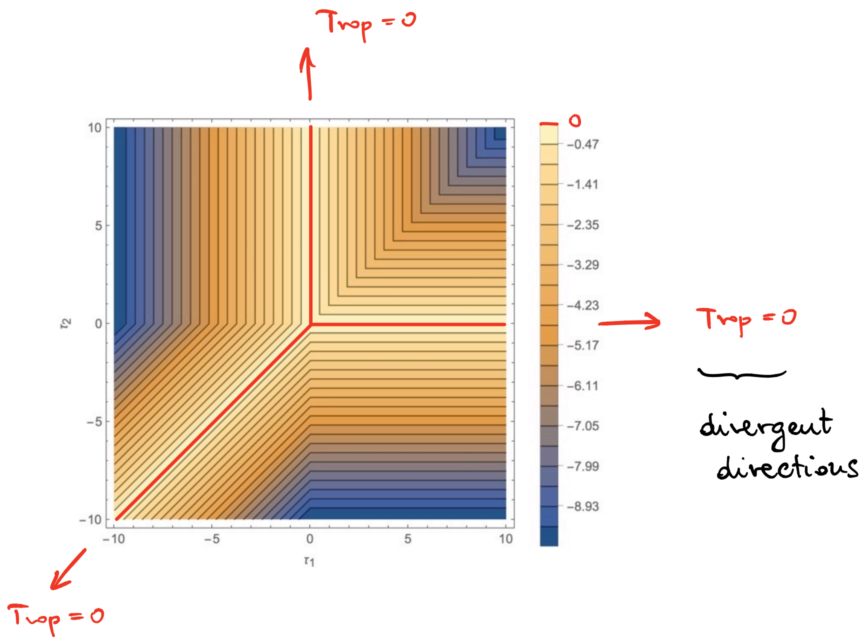
This are just the tropical rules! We introduce:

$$\text{Trop} := \tau_1+\tau_2+\tau_3 - 3\max(\tau_1+\tau_2,\ \tau_2+\tau_3,\ \tau_3+\tau_1) + \max(\ldots)$$

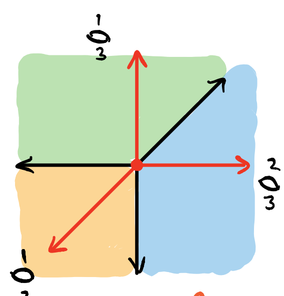
which has the same UV/IR behavior as

$$\frac{\alpha_1\alpha_2\alpha_3}{\mathcal{U}^3 \mathcal{F}^{-1}} \longrightarrow e^{\text{Trop}} + \ldots$$

Because of the max conditions, Trop is piecewise linear:

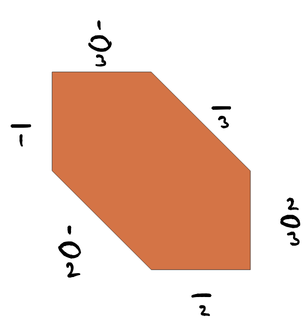


Normal fan:



domains of linearity
(share the same symmetry)

Dual polytope:



E.g. the $\mathcal{U}V$ bubble sub-divergence gives

$$-m_3^2 \underbrace{\int_0^{\cdots} d\lambda\, \lambda^{\epsilon-1}}_{1/\epsilon} \underbrace{\int_0^{\infty} \frac{d\alpha_1}{(1+\alpha_1)^2}}_{1} + \text{perm.}$$

connection to volumes of polytopes, intersection theory etc.

$$= -\frac{1}{\epsilon}\left(m_1^2 + m_2^2 + m_3^2\right)$$

For a general Feynman integral,

$$I = \int \frac{d^{E-1}\alpha}{\prod_e \alpha_e} \frac{N \prod_e \alpha_e}{\mathcal{U}^{D/2-\gamma} F^{\gamma}} \left(\frac{\mathcal{U}^{L+1}}{F^L}\right)^{\epsilon}$$

we introduce the Trop function

$$\text{Trop} := \sum_e \tau_e + \underbrace{\max(\ldots)}_{\text{monomials in } N} - \left(\tfrac{D}{2}-\gamma\right) \underbrace{\max_T(\ldots)}_{\text{spanning trees}} - \gamma \underbrace{\max_{T_L \cup T_R}(\ldots)}_{\text{spanning 2-trees}}$$

→ If Trop ≤ 0 everywhere, the integral has at most log divergences

→ Trop = 0 determines the scaling of Schwinger parameters

→ Works for UV/IR, any masses, planarity, ...

→ The normal fan determines how divergences fit together

→ Leading coefficient related to positive geometries

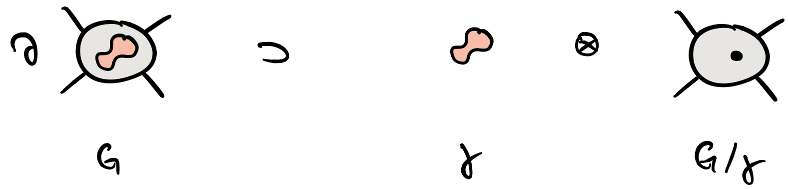
$$I = \frac{1}{\varepsilon^{\dim\{\mathrm{Trop}=0\}}} \underbrace{\int_{\substack{\text{adjacent}\\ \text{cones}}} e^{\mathrm{Trop}} (\dots)^{\epsilon}\, d^{E-1}\tau}_{\mathrm{vol}\,\{\mathrm{Trop}=0\}^{\vee} \times \text{ coefficient}} + \dots$$

→ The dual polytope can be defined in terms of Minkowski sums / differences of Newton polytopes

$$\mathbf{U} := \mathrm{Newt}(\mathcal{U})$$

$$\mathbf{F} := \mathrm{Newt}(\mathcal{F})$$

have known UV factorization properties

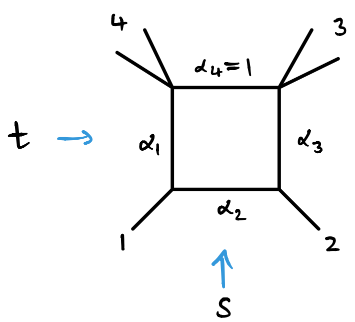


(with some restrictions on allowed γ, G/γ)

The combination $\sqcup \oplus \mathbf{F}$ was studied by Schultka as a realization of the Bloch-Esnault-Kreimer-Brown blow-up (connections to generalized permutohedra, ...)

What about the IR?

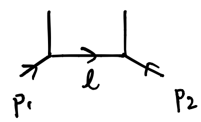
IR Example



all $m_e = 0$

$E = 4$
$L = 1$
$D = 4 - 2\epsilon$

Soft-collinear singularity



$$\ell^2 = \ell \cdot p_1 = \ell \cdot p_2 = 0$$

from which we expect the integral to behave $\sim \dfrac{1}{\epsilon^2}$.

The Feynman integral is

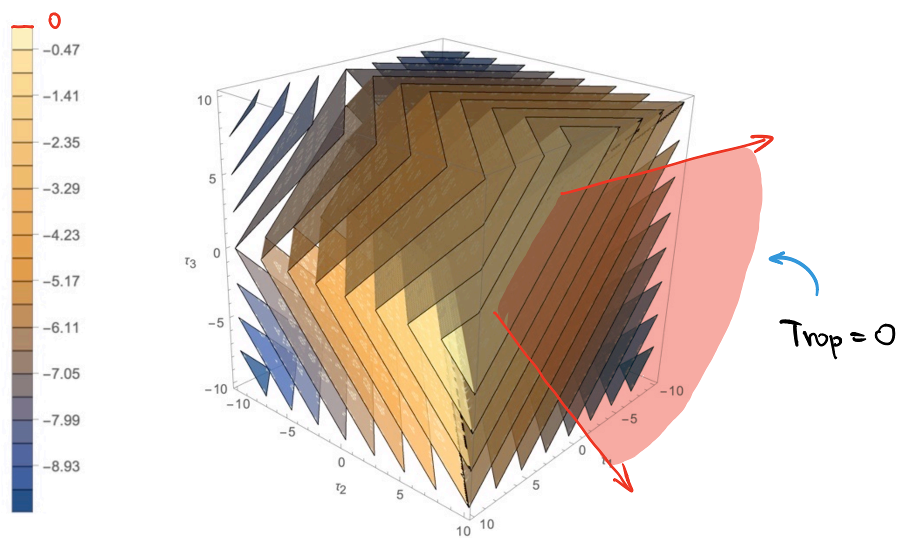
$$\Gamma(2+\epsilon) \int \frac{d^3\alpha}{\alpha_1\alpha_2\alpha_3\alpha_4} \frac{\alpha_1\alpha_2\alpha_3\alpha_4}{F^2} \left(\frac{\mathcal{U}^2}{F}\right)^{\epsilon} \Bigg|_{\alpha_4=1}$$

with $\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$

$$F = s\,\alpha_1\alpha_3 + t\,\alpha_2\alpha_4 + M_3^2\,\alpha_3\alpha_4 + M_4^2\,\alpha_4\alpha_1 .$$

Fixing $\alpha_4 = e^{\tau_4} = 1$, the Trop function is given by

$$\text{Trop} := \tau_1+\tau_2+\tau_3 - 2\max(\tau_1+\tau_3,\ \tau_2,\ \tau_3,\ \tau_1)$$



Divergences can be found by solving Trop = 0:

$$\{\tau_2 = \tau_1+\tau_3,\quad \tau_1 \geq 0,\quad \tau_3 \geq 0\} \rightsquigarrow \text{2-dim}$$

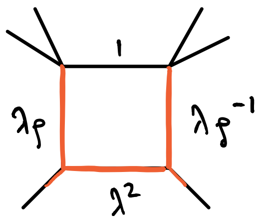
Verify:

$$\text{Trop}\big|_{\tau_2=\tau_1+\tau_3} = 2(\tau_1+\tau_3) - 2\max(\underbrace{\tau_1+\tau_3,\ \overset{\tau_2}{\overset{\parallel}{\tau_1+\tau_3}}}_{\text{dominate}},\ \tau_3,\ \tau_1) = 0$$

In other words:

$$\frac{\alpha_1\alpha_2\alpha_3}{F^2} \rightarrow \frac{\alpha_1\alpha_2\alpha_3}{(s\alpha_1\alpha_3 + t\alpha_2)^2} + \dots$$

Identify the symmetry:



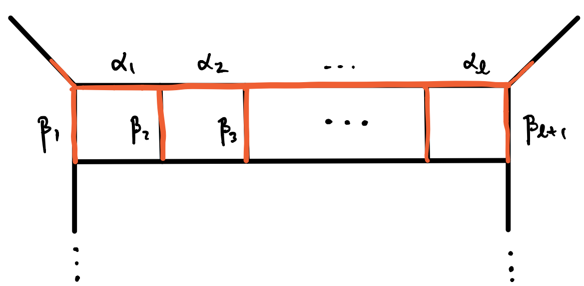
$$\alpha_1 \to \lambda \rho \, \alpha_1$$
$$\alpha_2 \to \lambda^2 \alpha_2$$
$$\alpha_3 \to \lambda \rho^{-1} \alpha_3$$

with $\lambda \to \infty, \quad \rho \in \mathbb{R}$

[cf. Yelleshpur '19]

Coefficient of the divergence is $\dfrac{1}{\epsilon^2} \dfrac{1}{st}$.

This is the first example of a comb



comb with ℓ loops

→ Overall IR scaling $\alpha_i \sim \Lambda^{\ell+1}$, $\beta_i \sim \Lambda^{\ell}$

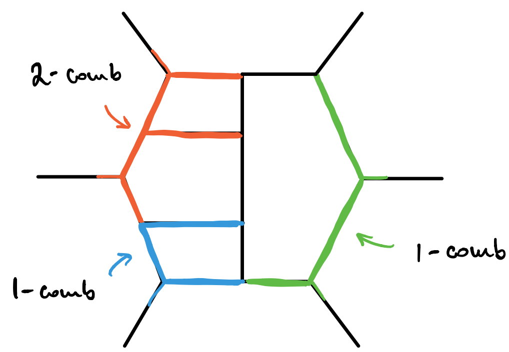
→ Each subcomb $\alpha_i > \beta_i, \beta_{i+1}$

leads to a $\sim \log^{2\ell}$ or $\sim 1/\epsilon^{2\ell}$

soft-collinear divergence

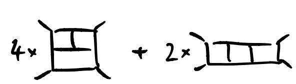
Comb-inatorics

Comb-atibility rules for fitting combs into a single diagram



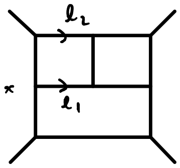
Consistency check : log of the $\mathcal{N}=4$ MHV amplitude

$$\log A_4^{\text{full}} = g\, A_4^{\text{1-loop}} + g^2 \left(A_4^{\text{2-loop}} - \frac{1}{2} \left(A_4^{\text{1-loop}}\right)^2 \right)$$
$$+ g^3 \left(A_4^{\text{3-loop}} - A_4^{\text{2-loop}} A_4^{\text{1-loop}} + \frac{1}{3} \left(A_4^{\text{1-loop}}\right)^3 \right) + \dots$$



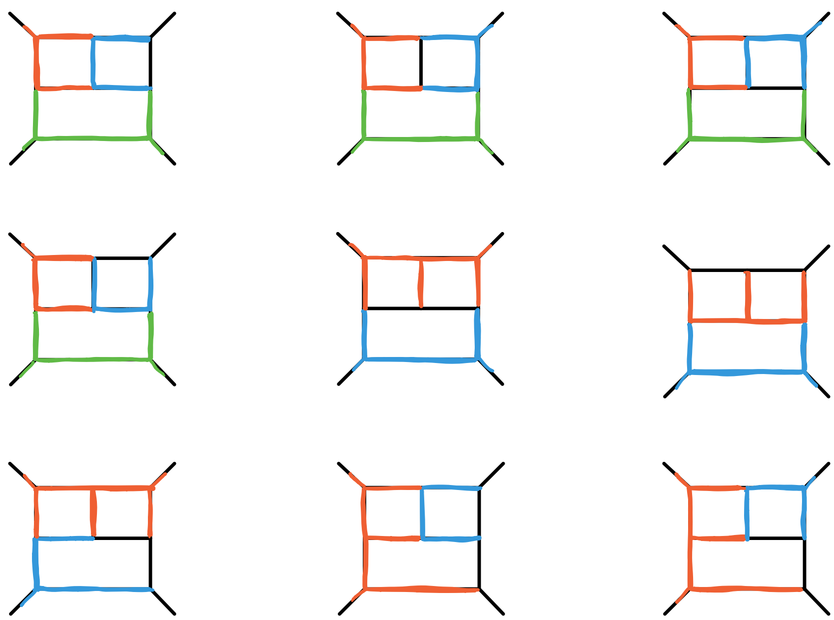
is supposed to have at most $\log^2$ IR singularity.

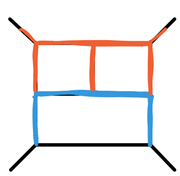
For example, the tennis court diagram is

$(\ell_1+\ell_2)^2 \times$ 

$$= \int \frac{d^{10}\alpha}{GL(1)} \, \frac{\mathcal{U}\tilde{\mathcal{F}} - \tilde{\mathcal{U}}\mathcal{F}}{\mathcal{U}^{-2}\mathcal{F}^{4}} \left(\frac{\mathcal{U}^4}{\mathcal{F}^3}\right)^{\epsilon}$$

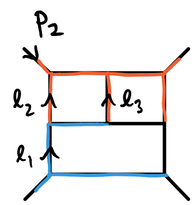
The leading divergence is dictated by the following comb-ination of soft-collinear limits:





$$\sim \frac{16}{9\epsilon^6} \, \frac{1}{s t^2} + \dots$$

As an aside, we're found an example of a quadratic IR divergence for the tennis court with no numerator:



→ Coming from $l_1, l_2, l_3 \propto p_2$

→ Diagnose with $\mathrm{Trop} > 0$

Summary

→ Tropical geometry is a realization of the idea of separation of scales

→ $\mathrm{Trop} = 0$ determines the directions in which the Feynman diagram has UV/ IR divergences

→ The soft-collinear divergences are governed by combs

Thanks!