Stasheff Polytopes, ~~Cosmology~~, and the Ising Model

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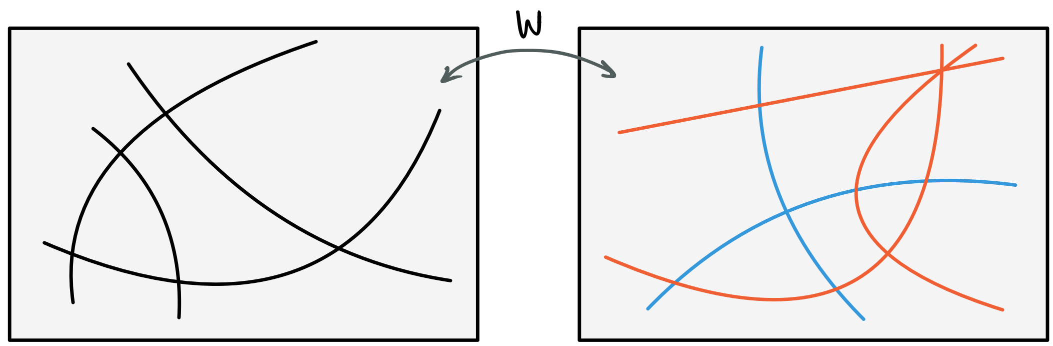
based on work with Ruth Britto, Lorenz Eberhardt, Shota Komatsu, Si Li, Andrzej Pokraka, Carlos Rodriguez, and Oliver Schlotterer

Common geometric picture for many physical quantities:



physical space | configuration space

Linked by a potential W. This structure is associated to the buzzwords: twisted cohomology, intersection theory, scattering equations, …

Quantity	Physical space	Configuration space	Parameter
tree-level amplitudes (loop integrands) (celestial)	Mink_4 kinematic space	moduli space $\mathcal{M}_{0,n}$ ($\mathcal{M}_{0,n+2g}$)	α'
Feynman integrals	Mink_4 kinematic space	loop momentum space / moduli space of worldlines	ε
AdS correlators / cosmological wavefunctions	conformal generators	moduli space $\mathcal{M}_{0,n}$	$\dfrac{\ell_{\text{string}}}{\ell_{(A)dS}}$

Landau-Ginzburg/ Seiberg-Witten theories	moduli space of vacua	$\mathbb{C}^4$	$1/\hbar$
⋮	⋮	⋮	⋮
?	?	$G(k,n)/T$ binary geometries	?
⋮	⋮	⋮	⋮
CFT_2 correlators	charges & positions of operators	configuration space $C_{n,k}$	$1/p$

this talk

Common feature: moduli space localization

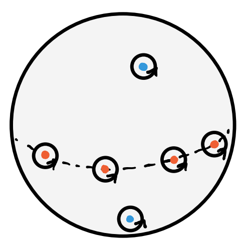
new residue theorems

{ zeros of dW } $\longleftrightarrow$ { poles of dW }

"scattering equations"

"boundaries"

E.g. on $\mathbb{CP}^1$



zeros = # poles -2

by Riemann-Roch

for logarithmic dW

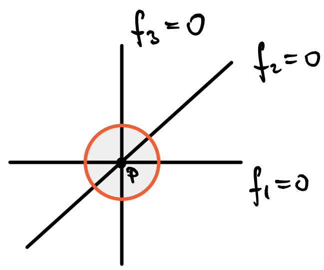
Plan for the talk:

- → Old and new residue theorems
- → Combinatorics and topology of configuration spaces
- → Correlation functions of minimal models & uniform transcendentality

Residues

Say we have $\mathbb{CP}^m - \bigcup_{i=1}^{m} \{ f_i = 0 \}$. There are many ways to define what "residue" means:

m hypersurfaces



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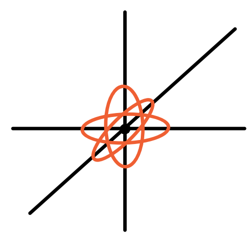


S^{2m-1} $\qquad\qquad$ $(S^1)^m$

Poincaré residue $\qquad\qquad$ Grothendieck residue

(NB whatever cartoon we draw in $m > 1$, it should be taken with a grain of salt)

$$\eta_\varphi = \# \frac{\langle \bar{f}\, d^{m-1} \bar{f} \rangle}{\| f \|^{2m}}\, g(z)\, d^m z \quad \longleftrightarrow \quad \varphi = \frac{g(z)\, d^m z}{f_1 f_2 \cdots f_m}$$

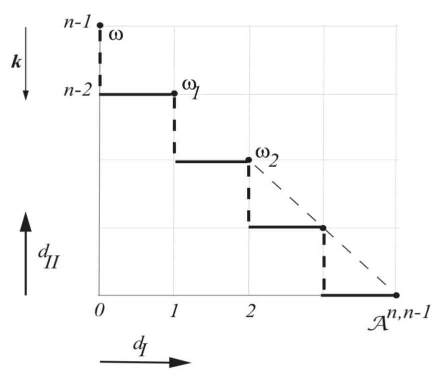
$(m, m-1)$ - form $\qquad\qquad$ $(m, 0)$ - form

Such that

definition

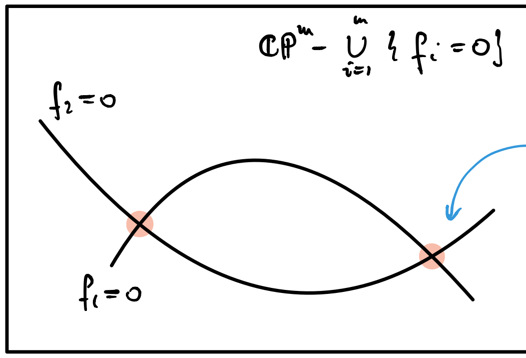
$$\oint_{S^{2m-1}} \eta_\varphi = \oint_{(S^1)^m} \varphi =: \mathrm{Res}_p(\varphi)$$

Proof of this fact is extremely illuminating, see, e.g.,



[Nicolaescu, notes from the "Absolutely fabulous seminar in algebraic geometry"]

Global residue theorem:



$\mathcal{P} = \bigcap_{i=1}^m \{f_i = 0\}$

(assume all exceptional divisors are blown-up)

$\Leftrightarrow$ m f_i's meet at a point

$$\sum_{p \in \mathcal{P}} \mathrm{Res}_p(\varphi) = \sum_{p \in \mathcal{P}} \oint_{\partial B_p} \eta_\varphi = \pm \int_{\mathbb{CP}^m - \bigcup_{p \in \mathcal{P}} B_p} d\eta_\varphi = 0 .$$

ball around p

However, it's tricky to use in physical applications because it requires m hypersurfaces $\{f_i = 0\}$.

We can do better.

The idea is to package the information about boundaries into a potential W:

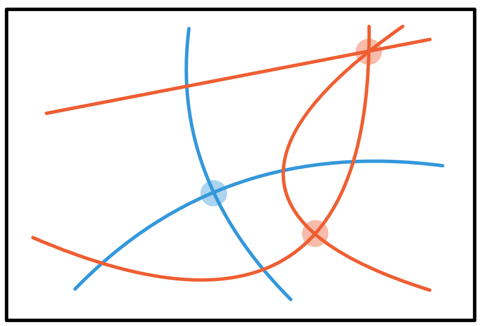
$$W = \sum_{i=1}^{k} \alpha_i \log f_i$$

whatever

weights (physics)

zeros of dW

$$\mathcal{Z} = \bigcap_{i=1}^{m} \{ \partial_{z_i} W = 0 \}$$



poles of dW

$$\mathcal{P} = \bigcap_{i=1}^{m} \{ f_i = 0 \}.$$

To obtain the new residue theorem we once again trace the zig-zag:

(0,0)-form

$$\nabla^{-1}\varphi_R$$

$$\begin{tikzcd}
C^0(\mathfrak{U},\Omega^m_M) & & & \\
C^0(\mathfrak{U},\Omega^{m-1}_M) \arrow[u, "\nabla_{dW}"] \arrow[r, "\delta"] & C^1(\mathfrak{U},\Omega^{m-1}_M) & & \\
& C^1(\mathfrak{U},\Omega^{m-2}_M) \arrow[u, "\nabla_{dW}"] \arrow[r, "\delta"] & C^2(\mathfrak{U},\Omega^{m-2}_M) & \\
& & \ddots \arrow[u, "\nabla_{dW}"] \arrow[r, "\delta"] & C^{m-1}(\mathfrak{U},\Omega^1_M) \\
& & & C^{m-1}(\mathfrak{U},\mathcal{O}_M) \arrow[u, "\nabla_{dW}"]
\end{tikzcd}$$

(m,0)-form

$$\varphi_R$$

[SM, Pokraka 1910.11852]

This gives

$$\sum_{p \in Z} \mathrm{Res}_p \big(\underbrace{\varphi_L}_{(m,0)\text{-form}} \underbrace{\nabla^{-1} \varphi_R}_{\text{function}} \big) = \pm \sum_{p \in P} \mathrm{Res}_p \big(\varphi_L \nabla^{-1} \varphi_R \big).$$

$$=: \langle \varphi_L \mid \varphi_R \rangle .$$

definition

Here $\psi_p := \nabla^{-1} \varphi_R$ is obtained by solving (schematically)

$$\varphi_R = \nabla \psi_p = (\partial_{x_1} + \partial_{x_1} W) \cdots (\partial_{x_m} + \partial_{x_m} W)\, \psi_R$$

locally near each p.

Around boundaries (poles of dW):

$W = \alpha'(\dots)$

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \frac{\mathrm{Res}_p(\varphi_L)\, \mathrm{Res}_p(\varphi_R)}{\prod_{i=1}^{m} \mathrm{Res}_p(dW)} + \mathcal{O}(\alpha')$$

for $\mathcal{M}_{0,n}$ this is expansion in terms of Feynman diagrams

Around critical points (zeros of dW):

$$\langle \varphi_L | \varphi_R \rangle = \pm \sum_{p \in \mathcal{Z}} \mathrm{Res}_p \left[\frac{\varphi_L\, \varphi_R}{\prod_{i=1}^{m} \partial_{z_i} W} \left(1 + \frac{1}{2\alpha'} \sum_{j=1}^{m} \frac{\partial_{z_j} \log(\varphi_L/\varphi_R)}{\partial_{z_j} W} + \mathcal{O}(1/\alpha'^2) \right) \right]$$

for $\mathcal{M}_{0,n}$ this is the CHY formula

- $\langle \varphi_L | \varphi_R \rangle$ is called the intersection number of φ_L and φ_R
- When φ_L and φ_R are logarithmic, the $\mathcal{O}(\alpha')$ and $\mathcal{O}(1/\alpha')$ corrections are absent
- In most physical applications φ_L and φ_R are not logarithmic, but often we can find logarithmic bases.

CFT$_2$ Correlators I

We'll consider minimal models labelled by (p, p') coprime, $p > p'$

$$S = \int_{S^2} d^2x \sqrt{g} \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{i}{\sqrt{2}} Q_{p,p'} \phi R \right).$$

complex action

Background charge: $Q_{p,p'} = \frac{p-p'}{\sqrt{pp'}}$.

Central charge: $c = 1 - 6 Q_{p,p'}^2$.

For example: $(p, p') = (3, 2)$ trivial CFT

$(p, p') = (4, 3)$ critical Ising model

$(p, p') = (5, 2)$ Yang-Lee edge singularity

$(p, p') = (5, 4)$ tricritical Ising model

$\vdots$

Operators $\mathcal{O}_q(x) = e^{i\sqrt{2}\, q\, \phi(x)}$ (charge) are classified by two integers (r, s):

$$1 \leq r \leq p' - 1 \qquad 1 \leq s \leq p - 1$$

Charges $q_{(r,s)}$ and **conformal dimensions** $h_{(r,s)}$:

$$q_{(r,s)} = \frac{p(1-r) - p'(1-s)}{2\sqrt{pp'}} , \qquad h_{(r,s)} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} .$$

Operators with (r,s) and $(p'-r,\ p-s)$ are identified.

e.g. $(p,p') = (4,3)$: **critical Ising model**, $Q_{4,3} = \frac{1}{2\sqrt{3}}$

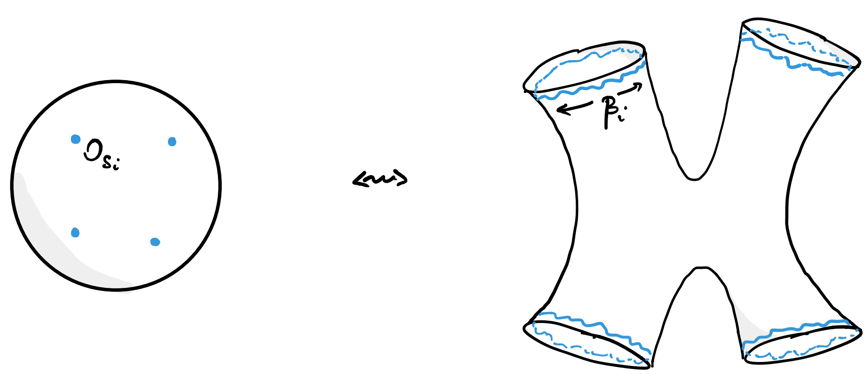
→ identity $\quad \mathbb{1} = \mathcal{O}_0 = \mathcal{O}_{1/2\sqrt{3}} \quad h = 0$

→ spin $\quad \sigma = \mathcal{O}_{\sqrt{3}/4} = \mathcal{O}_{-1/4\sqrt{3}} \quad h = 1/16$

→ energy $\quad \varepsilon = \mathcal{O}_{\sqrt{3}/2} = \mathcal{O}_{-1/\sqrt{3}} \quad h = 1/2$

Large-p limit of $(p,2)$ models coupled to **Liouville theory** is conjecturally describing **JT gravity**,

[Saad, Shenker, Stanford '19]



Charges labeled by $(1, s_i)$:

$$\mathcal{O}_{(1,1)}\ ,\quad \mathcal{O}_{(1,2)}\ ,\quad \mathcal{O}_{(1,3)}\ ,\quad \dots\ ,\quad \mathcal{O}_{(1,p-1)}$$

So the intuition might be (thanks Lorenz)

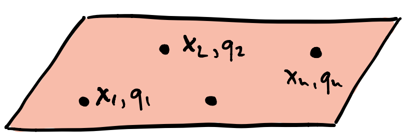
$$s_i \overset{?}{\rightsquigarrow} \beta_i$$

We will not consider coupling to Liouville.

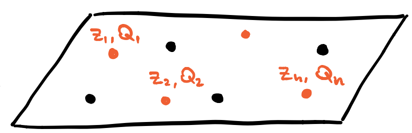
Coulomb gas formalism

[Dotsenko, Fateev '84]

[Kapec, Mahajan '20]



$$\rightsquigarrow \quad \int \prod_{i=1}^{k} d^2 z_i$$



replace background charge
with k screening operators

Neutrality condition
$$\sum_{i=1}^{n} q_i + \sum_{i=1}^{k} Q_i = Q_{p,p'}$$

– charge at infinity

$Q_i = \sqrt{p/p'}$ or $-\sqrt{p'/p}$ (fixed)

Hence any correlation function can be written as

$$\langle \mathcal{O}_{q_1}(x_1)\, \mathcal{O}_{q_2}(x_2) \cdots \mathcal{O}_{q_n}(x_n) \rangle_{MM} = \int \langle \mathcal{O}_{q_1} \cdots \mathcal{O}_{q_n} \mathcal{O}_{Q_1} \cdots \mathcal{O}_{Q_k} \rangle_{free} \prod_{i=1}^{k} d^2 z_i$$

$$= \int e^{W+\bar{W}} \prod_{i=1}^{k} d^2 z_i$$

with the potential

$$W = 2\sum_{i<j} Q_i Q_j \log(z_i - z_j) + 2\sum_{i,j} Q_i q_j \log(z_i - x_j) + 2\sum_{i<j} q_i q_j \log(x_i - x_j).$$

Physical space parametrized by q_i's and x_i's

Configuration space parametrized by z_i's.

Before diving into details, let's remind ourselves about the combinatorics of the above configuration space.

Configuration Spaces

$$C_{n,k} = \mathrm{Conf}_k\left(\mathbb{CP}^1 - \{n \text{ points}\}\right).$$

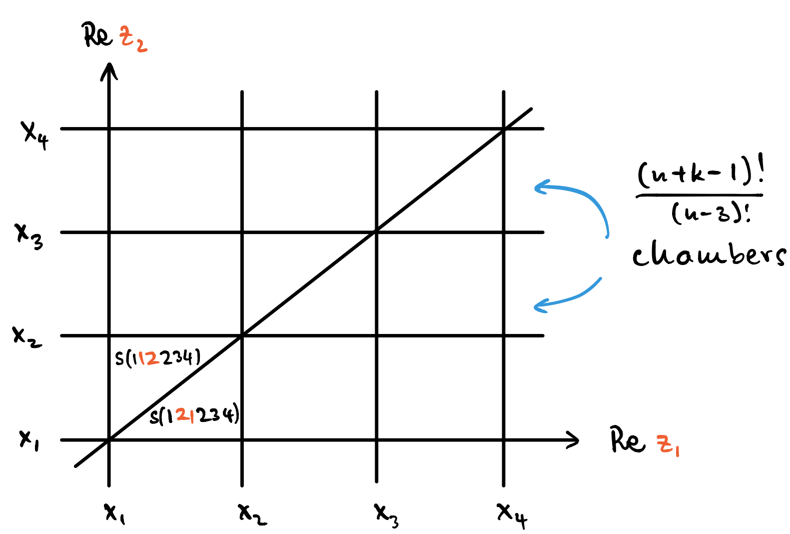
operators

screening charges

$$= (\mathbb{CP}^1)^k - \{z_i \neq z_j, x_j\}.$$

fix $x_j \in \mathbb{R}$

For example, $C_{4,2} = (\mathbb{CP}^1)^2 - \{z_1, z_2 \neq x_1, x_2, x_3, x_4\}$
$\qquad - \{z_1 \neq z_2\}$



Each chamber is combinatorially a product of Stasheff polytopes (AKA associahedra).

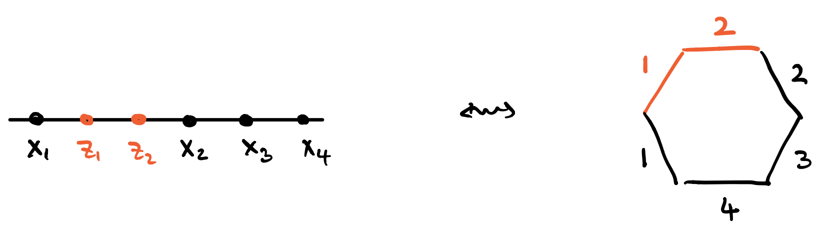
We know this structure from tree-level amplitudes, double-copy, etc., but now it has a small twist

[Carrasco '17]

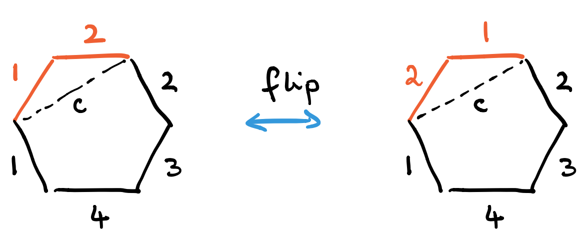
[SM 1706.08527]

[Arkani-Hamed et al. '17]

Bijection with colored $(n+k)$-gons, e.g.



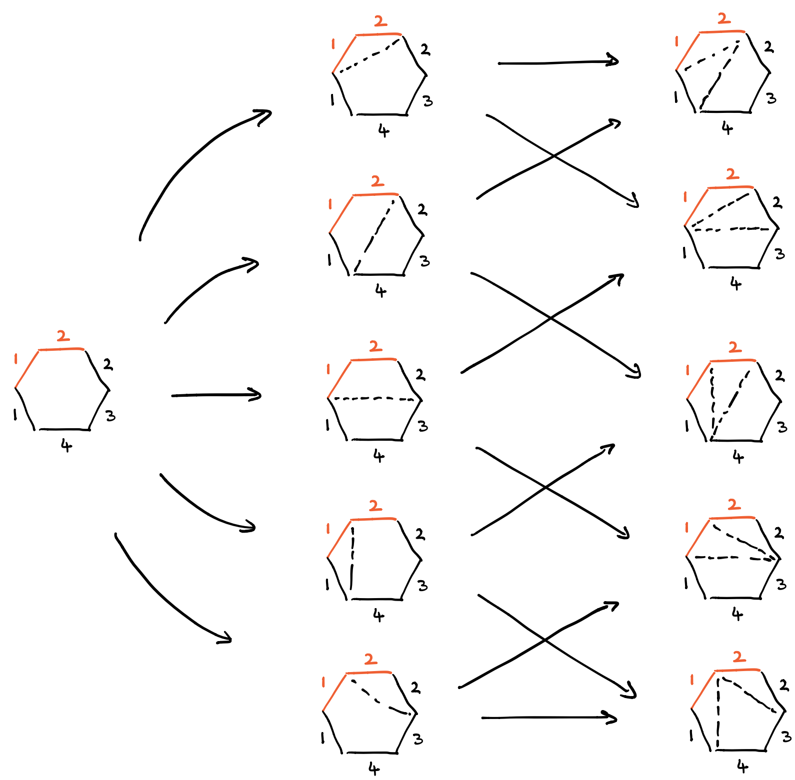
Adjacency structure of the Stasheff polytopes is described by tessellations of colored polygons



associated value (used later)

$v_c = 2Q_1Q_2$

A chord c is admissible if and only if it doesn't change the order of the fixed (black) edges. This defined the boundary structure:



Now we can compute intersection numbers of Stasheff polytopes, familiar from the KLT relations.

Let's just flash a result for self-intersection:

$$[S(\alpha) \mid S(\alpha)] = \sum_{\substack{\text{tessellations} \\ T}} \prod_{\substack{\text{chords} \\ c \in T}} \frac{1}{e^{2\pi i V_c} - 1} .$$

We can also define dual cycles such that

$$[S(\alpha) \mid S^\vee(\beta)] = \delta_{\alpha\beta}.$$

[Details in 2102.xxxxx with Britto, Rodriguez, Schlotterer]

How many integration cycles / differential forms are needed? The answer turns out to be a topological invariant of the configuration space:

$$|\chi(C_{n,k})| = \frac{(n+k-3)!}{(n-3)!}$$

Euler characteristic

Bases of logarithmic differential forms are given by "Parke-Taylor" factors:

canonical form

$$\mathrm{PT}(\cdots \rho_0 \rho_1 \rho_2 \cdots \rho_m \rho_{m+1} \cdots) = \prod \frac{x_{\rho_0} - x_{\rho_{m+1}}}{(x_{\rho_0} - z_{\rho_1})(z_{\rho_1} - z_{\rho_2}) \cdots (z_{\rho_m} - x_{\rho_{m+1}})} d^k z$$

$$\mathrm{PT}^\vee(\cdots \rho_0 \rho_1 \rho_2 \cdots \rho_m \rho_{m+1} \cdots) = \prod_i \sum_j \frac{Q_{\rho_i} q_{\rho_j}}{z_{\rho_i} - x_{\rho_j}} d^k z$$

Orthogonality: $\langle PT(\alpha) \mid PT^{\vee}(\beta) \rangle = \delta_{\alpha\beta}$.

check e.g. using the HomotopyContinuation.jl package

[see Simon Telen's talk]

CFT_2 Correlators II

Back to physics. Recall

$$\langle \mathcal{O}_{q_1}(x_1)\,\mathcal{O}_{q_2}(x_2)\cdots\mathcal{O}_{q_n}(x_n)\rangle_{MM} = \int_{C_{n,k}} e^{W+\overline{W}} \prod_{i=1}^{k} d^2 z_i$$

Every correlation function can be expanded as

$$\int_{C_{n,k}} e^{W+\overline{W}}\, d^k z \wedge d^k \bar{z} = \sum_{\rho,\tau} \underbrace{\langle d^k z \,|\, PT^\vee(\rho)\rangle\, \overline{\langle d^k z \,|\, PT^\vee(\tau)\rangle}}_{\text{rational coefficients}}$$

$$\times \underbrace{\int e^{\omega+\overline{\omega}}\, PT(\rho) \wedge \overline{PT(\tau)}}_{\text{basis integrals}}$$

As well as

$$\int_{C_{n,k}} e^{W+\overline{W}}\, \varphi_L \wedge \overline{\varphi_R} = \sum_{\alpha,\beta} \underbrace{[\, S^\vee(\alpha) \,|\, S^\vee(\beta)\,]}_{\text{inverse matrix of } [\, S(\alpha) \,|\, S(\beta)\,]} \underbrace{\int_{S(\alpha)} e^{W} \varphi_L}_{\text{"conformal blocks"}}\ \overline{\int_{S(\beta)} e^{W} \varphi_R}$$

Leading-order expansion:

$$\int e^{W+\overline{W}}\, PT(\alpha) \wedge \overline{PT^\vee(\beta)} = \langle PT(\alpha) \,|\, PT^\vee(\beta)\rangle + \ldots = \delta_{\alpha\beta} + \ldots$$

$$\int_{S(\alpha)} e^{W}\, PT^\vee(\beta) = \langle PT(\alpha) \,|\, PT^\vee(\beta)\rangle + \ldots = \delta_{\alpha\beta} + \ldots$$

Number theory magic: [$k=1$, $n=3$ were known]

$$\int e^{W+\overline{W}}\, PT(\rho)\wedge\overline{PT(\tau)} = \mathrm{sv}\left(\int_{S(\tau)} e^{W}\, PT(\rho)\right).$$

single-valued map, e.g., $\mathrm{sv}(\log z) = \log|z|^2$ etc.

Galois coaction:

$$\Delta\left(\int_{S(\alpha)} e^{W}\, P\check{T}(\beta)\right) = \sum_{\gamma}\left(\int_{S(\alpha)} e^{W}\, P\check{T}(\gamma)\right)\otimes\left(\int_{S(\gamma)} e^{W}\, P\check{T}(\beta)\right).$$

Monodromies:

$$\int e^{W(x)}\, P\check{T}(\beta) = \mathcal{P}\exp\int_{y}^{x}\Omega \cdot \int e^{W(y)}\, P\check{T}(\beta).$$

braid matrices

Can be treated as generating functions for MPL's and their coaction / monodromy properties.

Let's look at an explicit example :

4-pt function of $(r,s)=(2,1)$ operators .

Requires a single screening charge $(k=1)$

$$\langle \mathcal{O}_{2,1}(0)\, \mathcal{O}_{2,1}(x)\, \mathcal{O}_{2,1}(1)\, \mathcal{O}_{2,1}(\infty) \rangle_{MM} = \# \int_{\mathbb{C}-\{0,x,1\}} d^2z \;\; |z|^{-2p/p'} \; |z-x|^{-2p/p'} \; |z-1|^{-2p/p'}$$

cross-ratio

Critical Ising model : $(p,p') = (4,3)$

$\mathcal{O}_{2,1} = \varepsilon$ energy operator

$$\langle \varepsilon(0)\, \varepsilon(x)\, \varepsilon(1)\, \varepsilon(\infty) \rangle_{\text{Ising}} = \# \left| \frac{1}{x} - \frac{1}{x-1} - 1 \right|^2$$

$$= \# \left| \operatorname{Pf}\left(\frac{1}{x_i - x_j} \right) \right|^2$$

free-fermion computation

Let's consider $(p,2)$ models in the large-p limit

For example, the 4-pt function of $(r,s)=(1,2)$ operators reads

$$\langle O_{1,2}(0)\, O_{1,2}(x)\, O_{1,2}(1)\, O_{1,2}(\infty)\rangle_{MM} = \# \int_{\mathbb{C}-\{0,x,1\}} d^2 z \; |z|^{-4/p}\, |z-x|^{-4/p}\, |z-1|^{-4/p}$$

Using the aforementioned techniques we can expand it in $1/p$ in terms of single-valued MPL's:

$$\begin{aligned} = \# \Big[& \left(1+|x|^2+|1-x|^2\right) \\ & - \frac{6}{p}\left(|x|^2 \log|x|^2 + |1-x|^2 \log|1-x|^2\right) \\ & + \frac{12}{p^2}\Big(2|x|^2\, \mathrm{sv}\, G(0,0;x) + x(\bar{x}-1)\, \mathrm{sv}\, G(0,1;x) \\ & \qquad + \bar{x}(x-1)\, \mathrm{sv}\, G(1,0;x) + 2|1-x|^2\, \mathrm{sv}\, G(1,1;x)\Big) \\ & + \ldots \Big] \end{aligned}$$

Assign transcendental weights $\mathcal{T}(1/p) = -1$, $\mathcal{T}(x) = 0$, $\mathcal{T}(\pi) = 1$, etc.

The correlation function is UT to all orders in $1/p$!

Summary

- We considered yet another example of physical observables that can be described in terms of the twisted cohomology formalism: CFT_2 correlation functions
- Interesting geometry: Stasheff polytopes, colored polygons, intersection theory, coaction ...
- First examples of uniform transcendentality appearing in CFT_2 correlators

Thanks!