Progress on Crossing Symmetry

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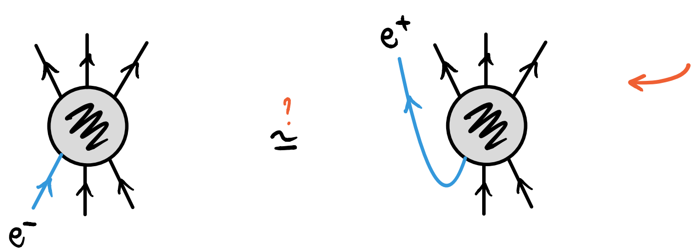
California Amplitudes Meeting 2021
March 25, 2021

What is crossing symmetry?

$p^\mu = (E, \vec{p})$ $\qquad$ $-p^\mu = (-E, -\vec{p})$

On the level of observables:



for any number and type of the remaining particles

Are scattering amplitudes in different crossing channels boundary values of the same function?

→ Proposed in 1954 by Gell-Mann, Goldberger, and Thirring

→ As of 2021 we still don't know if it's true or not

Let's start by reviewing what is known about crossing symmetry, really.

It's believed that the following physical assumptions will be needed in the proof:

- Locality (Existence of local operators $O(x)$)
- Causality ($[O(x), O(y)] = 0$ when $(x-y)^2 < 0$)
- Unitarity ($\mathbb{1} = \int d^4\tilde{p}\ |p\rangle\langle p|$)

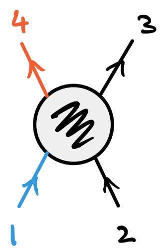
Still not enough! We also need:

- Mass gap (no massless particles)

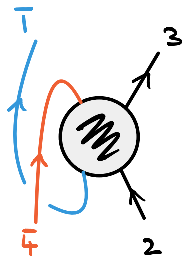
non-perturbatively

With these assumptions one can show crossing between:

$2 \to 2$:

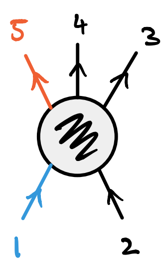


$\simeq$

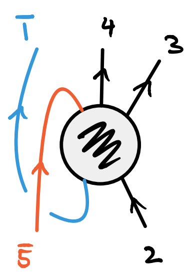


[Bros, Epstein, Glaser '65]

$2 \to 3$:



$\simeq$



[Bros '86]

The simplest cases where crossing symmetry has never been proven:

→ Any process involving massless particles

→ Different number of in/out states, e.g.,

$$12 \to 34 \quad \text{and} \quad 1 \to \bar{2}34$$

$$12 \to 345 \quad \text{and} \quad 12\bar{3} \to 45$$

→ Any amplitude with $n > 5$ external particles

→ Warning: many incorrect statements about crossing in the massive gravity EFT literature

So why don't we teach it in QFT courses?

→ Almost entirely complex analysis theorems

→ No physical understanding

The purpose of this talk is to study crossing symmetry in a simplified setup and propose what its physical interpretation might eventually entail.

The idea (due to Witten) is to prove crossing symmetry on-shell within the framework of perturbation theory, where one might reasonably hope to circumvent the aforementioned issues.

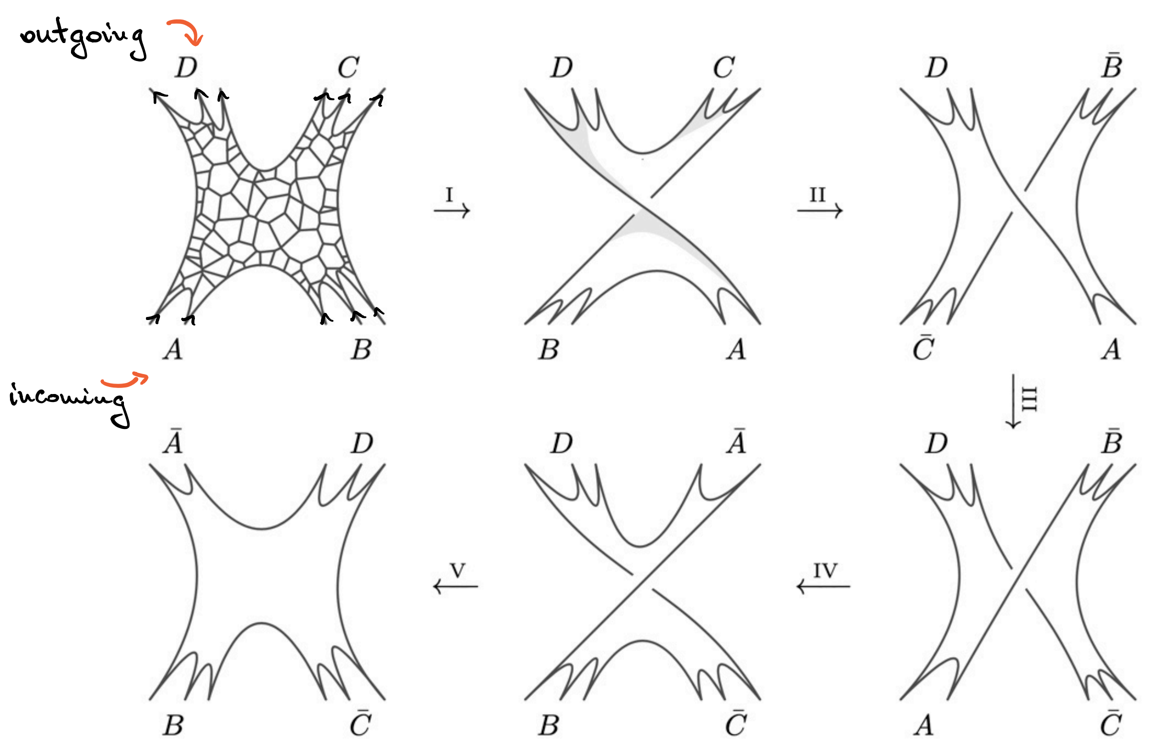
- → Singularities have a clear physical meaning and are governed by Landau equations
- → Standard ways of dealing with UV/IR divergences
- → Same Feynman rules for any multiplicity
- → This talk: planar amplitudes only
  (all loops, multiplicities, masses, ...)

We will show that if an amplitude in the channel $AB \to CD$ exists, the crossed amplitude is given by its analytic continuation,

$$S_{AB\to CD} = S_{B\bar{C}\to D\bar{A}}$$

explaining what this means is the content of this talk

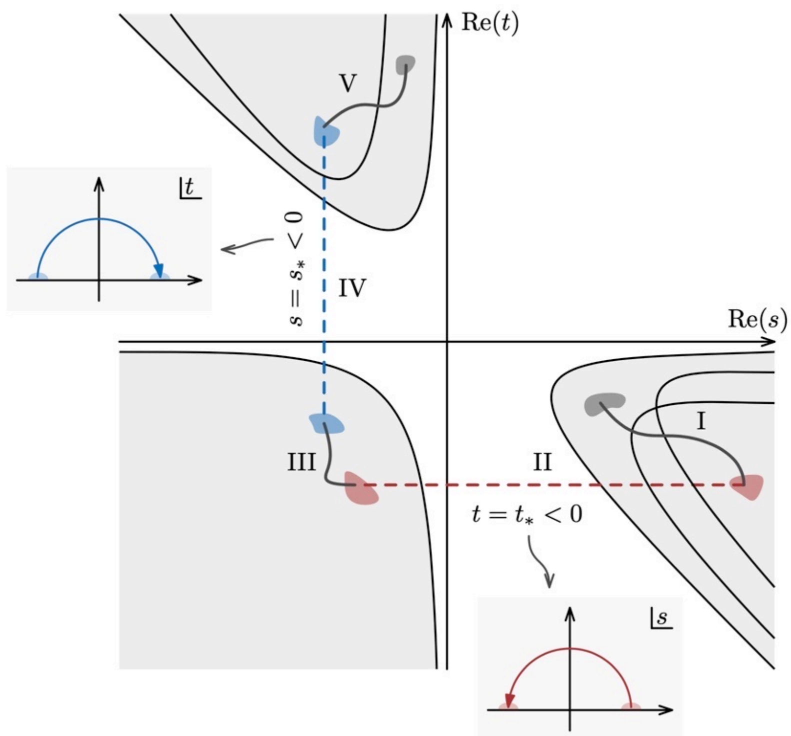
Before diving into details, let's illustrate this continuation on a cartoon level:



→ Each diagram represents a configuration that <u>cannot</u> have singularities

For concreteness we will focus on the simplest 4-pt example with $A=\{1\}$, $B=\{2\}$, $C=\{3\}$, $D=\{4\}$.

Two Mandelstam invariants $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$



It should be stressed that $n=4$ continuation can be formulated more simply, but this one

→ has physical interpretation

→ generalizes to arbitrary n

→ has a chance of working for non-planar amplitudes

Outline

1. Review of Landau equations

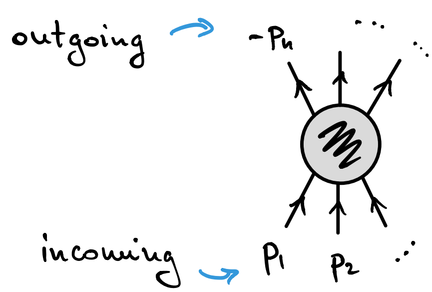
2. Energy flow in planar diagrams

3. Analytic continuation near physical regions (steps I, II, IV)

4. Analytic continuation in crossing domains (steps II, IV)

5. Putting everything together

Conventions:



→ momentum conservation reads

$\sum_{i=1}^{n} p_i^\mu = 0$ and masses $p_i^2 = M_i^2$.

→ $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

Review of Landau equations

In order to make the question of crossing symmetry well-posed we need to assume the amplitude exists in the first place.

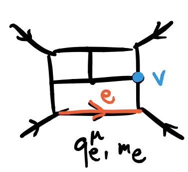
In other words, all the overall divergences have been
→ renormalized (e.g. BPHZ renorm.); and/or
→ regularized (e.g. analytic / dimensional reg.)

In a CPT-invariant quantum field theory admitting local Feynman rules, scattering amplitudes are linear combinations of Feynman integrals

$$I = \int d^{DL} \ell_a \, N \prod_{e=1}^{E} \frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon}$$

set N=1 for now (scalar)

D = space-time dimension
L = # loops
E = # internal edges (propagators)



Mom. cons. at every vertex v:

$$p_v^\mu + \sum_{e \ni v} \pm q_e^\mu = 0$$

ext' mom. orientation

(For non-local theories see [Chin, Tomboulis '18]
[de Lacroix, Erbin, Sen '18])

Introduce Schwinger parameters α_e :

$$\frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon} = \int_0^\infty d\alpha_e \; e^{\frac{i}{\hbar}(q_e^2 - m_e^2 + i\epsilon)\,\alpha_e}$$

convergence at ∞

Applying it to every propagator we obtain

$$I = \int d^{DL}\ell_a \; d^E\alpha_e \; e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e\right)}$$

where

$$V(\alpha_e, q_e^\mu) = \sum_{e=1}^{E} (q_e^2 - m_e^2)\,\alpha_e$$

In the classical limit, $\hbar \to 0$, they are dominated by saddle points; ignoring boundaries:

→ Vary ℓ_a^μ :
$$\sum_{e \ni \ell} \pm q_e^\mu \alpha_e = 0 .$$
orientation

particle going on-shell

→ Vary α_e: $q_e^2 - m_e^2 = 0$.

necessary cond^s for singularities

These are known as the leading Landau equations:

linear
$$\begin{cases} p_v^\mu + \sum_e \pm q_e^\mu = 0 & \forall \text{ vertices } v \\ \sum_e \pm q_e^\mu \alpha_e = 0 & \forall \text{ loops } a \end{cases}$$

quadratic
$$\begin{cases} q_e^2 - m_e^2 = 0 & \forall \text{ edges } e \end{cases}$$

There are also boundary saddle points

→ Subleading LE $(\alpha_e \to 0, \infty)$:

same as leading LE for a simpler diagram, but we already consider all (planar) diagrams

→ Second-type LE $(\ell_a^\mu \to \pm \infty)$:

only matter after performing large contour deformations (e.g. discontinuities) which we won't do (& only at special kinematic configurations)

At this stage the integral is Gaussian in the loop momenta, so we can just integrate them out:

$$I = \# \int_0^\infty \frac{d^E \alpha_e}{\mathcal{U}^{D/2}} \; N \; e^{\frac{i}{\hbar}\left(V + i\epsilon \sum_e \alpha_e\right)}.$$

↑ inclusion of spin interactions cannot introduce new singularities

where

$$V(\alpha_e) = \sum_{e=1}^{E} \left(q_e^2 - m_e^2\right) \alpha_e \Big|_{\text{linear LE}} \qquad q_e^2 = q_e^2(s_{ij}, \alpha_e)$$

which is only a function of Schwinger parameters and Mandelstam invariants.

Here the Hessian is the first Symanzik polynomial

$$\mathcal{U} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} \alpha_e > 0 .$$

NB the interpretation of Landau equations as saddle points isn't standard in the literature, but that's how we should think about them as physicists.

(can be formulated rigorously in terms of stratified Morse theory)

In fact, V is nothing more than the worldline action.

You often see the overall $GL(1)$ gauge $\alpha_e \to \lambda \alpha_e$ fixed to $\sum_{e=1}^{E} \alpha_e = 1$, but this would only obscure some features, so let's not do it.

Causality is imposed by inserting $i\epsilon$ "by hand". We will instead:

→ deform the contour

→ deform the external kinematics

to the same effect, i.e., $\operatorname{Im} V > 0$.

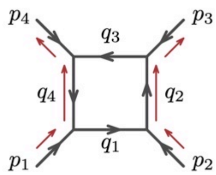
At this level, saddle-point equations read

$$\frac{\partial V}{\partial \alpha_e} = q_e^2 - m_e^2 = 0 .$$

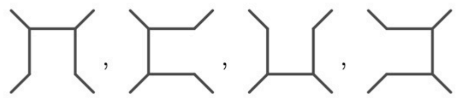
which also implies

$$V = \sum_{e=1}^{E} \alpha_e \frac{\partial V}{\partial \alpha_e} = 0 \qquad \text{by homogeneity.}$$

Simple example and energy flow in planar diagrams



Spanning trees

$T \in \{$  $\}$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$$

Linear Landau equations

$$p_i^\mu - q_i^\mu + q_{i-1}^\mu = 0 \qquad\qquad i = 1,2,3,4$$

$$\sum_{e=1}^{4} \alpha_e q_e^\mu = 0$$

have the solution

$$q_1^\mu = \frac{-p_2^\mu \alpha_2 - p_{23}^\mu \alpha_3 + p_1^\mu \alpha_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}, \qquad q_2^\mu = \frac{p_2^\mu \alpha_1 - p_3^\mu \alpha_3 + p_{12}^\mu \alpha_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$$

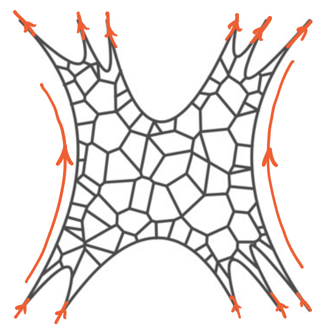
$$q_3^\mu = \frac{p_{23}^\mu \alpha_1 + p_3^\mu \alpha_2 + p_{123}^\mu \alpha_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}, \qquad q_4^\mu = \frac{-p_1^\mu \alpha_1 - p_{12}^\mu \alpha_2 - p_{123}^\mu \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}.$$

Let's look at the energy component $\mu = 0$ for $12 \to 34$ scattering, i.e., $p_1^0, p_2^0 > 0$ and $p_3^0, p_4^0 < 0$ with $\alpha_e > 0$:

$$q_2^0 > 0 \qquad \text{and} \qquad q_4^0 < 0$$

If there's a singularity, the energy can only flow in the causal direction along its sides!

This is a general fact. One explanation is that on the saddle point the total Lorentzian length of the diagram wants to be minimized:

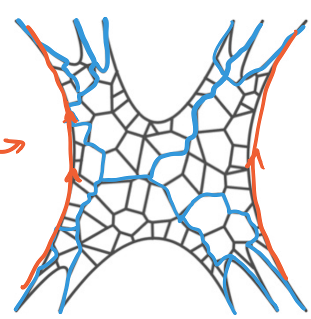


We can also look at the explicit solution of linear Landau equations:

$$q_e^\mu = \frac{1}{u} \sum_{\substack{\text{spanning} \\ \text{trees } T}} p^\mu_{e,T} \prod_{e' \notin T} \alpha_{e'}$$

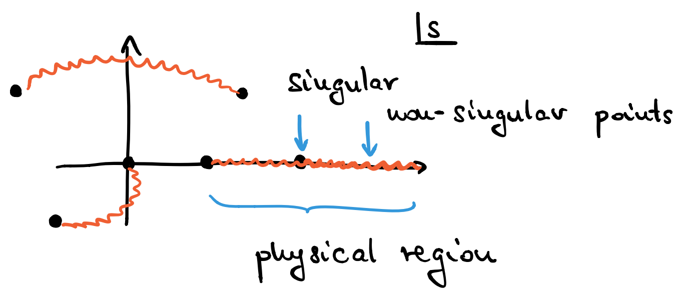
total external momentum flowing through e along T

impossible to draw a spanning tree where energy doesn't flow upwards



Analyticity near the physical regions

doesn't assume planarity



Near non-singular points we can simply deform the integration contour $\alpha_e \mapsto \check{\alpha}_e$. There exists a canonical deformation:

$$\check{\alpha}_e = \alpha_e \, e^{i\epsilon(q_e^2 - m_e^2)}$$

$$= \alpha_e + i\epsilon(q_e^2 - m_e^2)\alpha_e + \mathcal{O}(\epsilon^2)$$

The deformed action reads

$$\check{V}(\check{\alpha}_e) = V(\alpha_e) + i\epsilon \sum_{e=1}^{E} (q_e^2 - m_e^2)\, \alpha_e \underbrace{\frac{\partial V}{\partial \alpha_e}}_{q_e^2 - m_e^2} + \mathcal{O}(\epsilon^2)$$

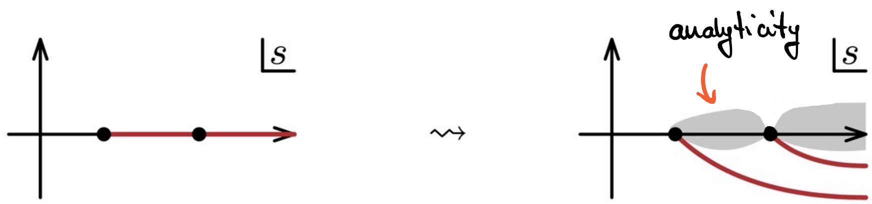
$$= V + i\epsilon \sum_{e=1}^{E} (q_e^2 - m_e^2)^2 \alpha_e + \mathcal{O}(\epsilon^2).$$

Its imaginary part is

$$\mathrm{Im}\,\check{V} = \epsilon \underbrace{\sum_{e=1}^{E} (q_e^2 - m_e^2)^2 \alpha_e}_{>0} + \dots$$

so for sufficiently small ϵ it implements the correct causality conditions.

Moreover, we can make small $O(\epsilon^2)$ deformations of external kinematics, which gives subleading corrections to V ⇒ Contour deformation still valid.



This doesn't work close to singular points where $q_e^2 - m_e^2 = 0$ for all e. Here the amplitude is only analytic from a certain direction. Deforming external kinematics:

$$\hat{p}_i^\mu = p_i^\mu + \epsilon^2 \Delta p_i^\mu$$

(s.t. it remains on-shell and momentum-conserving)

gives

$$\hat{q}_e^\mu = q_e^\mu + \epsilon^2 \Delta q_e^\mu .$$

For some (projective) solution of Landau equations α_e^* we have $(q_e^*)^2 - m_e^2 = 0$ and hence:

$$\hat{\alpha}_e^* = \alpha_e^* \, e^{i\epsilon((\hat{q}_e^*)^2 - m_e^2)}$$

deform external kin. & contour

$$= \alpha_e^* + \mathcal{O}(\epsilon^3).$$

This means

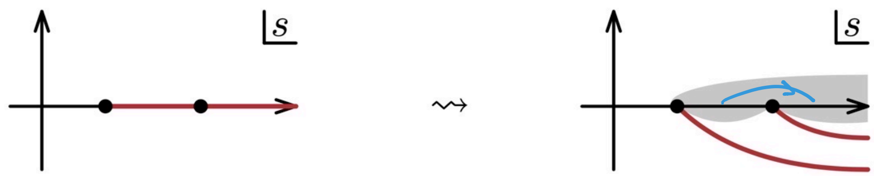
$$\hat{V}^* = \sum_{e=1}^{E} \left((\hat{q}_e^*)^2 - m_e^2 \right) \alpha_e^*$$

$$= 2\epsilon^2 \underbrace{\sum_{e=1}^{E} \Delta q_e^* \cdot q_e^* \, \alpha_e^*}_{\text{function of Mandelstam invariants only} \iff \text{definition of Landau curve}} + \mathcal{O}(\epsilon^3).$$

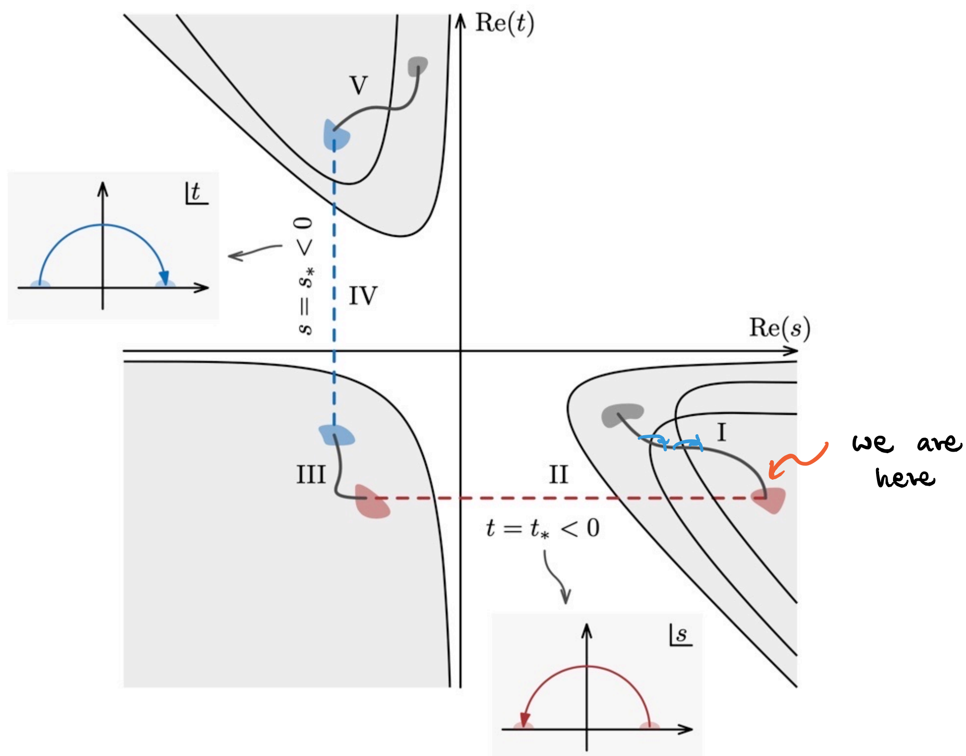
Now it's clear how to continue around this Landau singularity; go in

$$\mathrm{Im} \left(\sum_{e=1}^{E} \Delta q_e^* \cdot q_e^* \, \alpha_e^* \right) > 0$$

for sufficiently small ϵ.



s
$\rightsquigarrow$
s



Re(t)
V
t
$s = s_* < 0$
IV
Re(s)
I
we are here
III
II
$t = t_* < 0$
s

Analyticity in the crossing domains

Go to the Lorentz frame (in ligthcone coord^s)

$$ds^2 = dx^+ dx^- - d\vec{x}^2$$

$$p_1^\mu = (p_1^+ \, , \; p_1^- \, , \; \vec{p}_1)$$

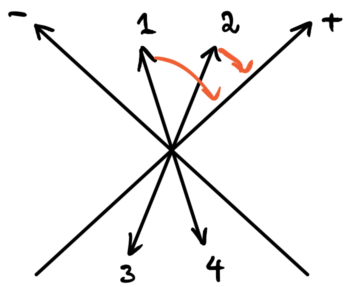
$$p_2^\mu = (p_2^+ \, , \; p_2^- \, , \; \vec{p}_2)$$

$$p_3^\mu = (-p_2^+ \, , \; -p_2^- \, , \; \vec{p}_3)$$

$$p_4^\mu = (-p_1^+ \, , \; -p_1^- \, , \; \vec{p}_4)$$

such that all particles are on-shell, $M_i^2 = p_i^+ p_i^- - \vec{p}_i^{\,2}$

and mom. cons. $\sum_i \vec{p}_i = 0$



Impose that 2 & 3 are closer to the +ve side of the lightcone,

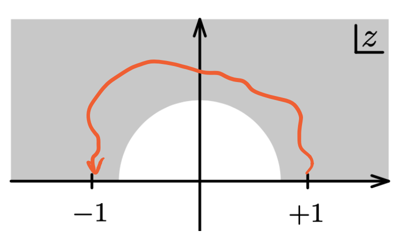
$$\frac{p_1^+}{p_1^-} < \frac{p_2^+}{p_2^-}$$

This defines the starting point of step II

Now let's rotate the energies:

$$\hat{p}_2^\mu = \left(z\, p_2^+ ,\ \frac{1}{z} p_2^- ,\ \vec{p}_2 \right)$$

$$\hat{p}_3^\mu = \left(-z\, p_2^+ ,\ -\frac{1}{z} p_2^- ,\ \vec{p}_3 \right)$$



In terms of Mandelstam invariants

$$\mathrm{Im}\, \hat{s} = \mathrm{Im}\,(p_1 + \hat{p}_2)^2$$

$$= \underbrace{\mathrm{Im}\, z}_{>0} \underbrace{\left(p_2^+ p_1^- - \frac{1}{|z|^2} p_1^+ p_2^- \right)}_{>0} > 0 .$$

$$\hat{t} = (\hat{p}_2 + \hat{p}_3)^2 = t \quad \text{fixed}$$

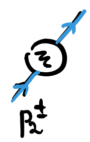
$$\Rightarrow \mathrm{Im}\, \hat{t} = 0$$

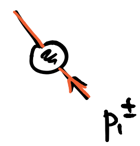
Hence in the imaginary direction it looks like highly energetic process in the forward limit (Regge^T)

We want to show Landau equations cannot have solutions along the path of deformation.

→ Linear LE:

$$\hat{q}_e^\pm = \underbrace{z^{\pm 1} p_2^\pm f_e} + \underbrace{p_1^\pm g_e}$$





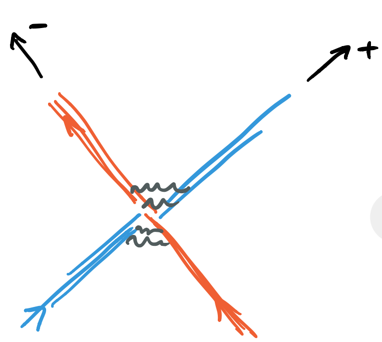
→ Quadratic LE:

$$\mathrm{Im}\left(\hat{q}_e^2 - m_e^2\right) = \mathrm{Im}\Big(\left(z\, p_2^+ f_e + p_1^+ g_e\right)\left(\tfrac{1}{z} p_2^- f_e + p_1^- g_e\right)\Big)$$

$$= \underbrace{\mathrm{Im}\, z}_{>0}\ \underbrace{f_e\, g_e}_{?}\ \underbrace{\left(p_2^+ p_1^- - \frac{1}{|z|^2} p_1^+ p_2^- \right)}_{>0}$$

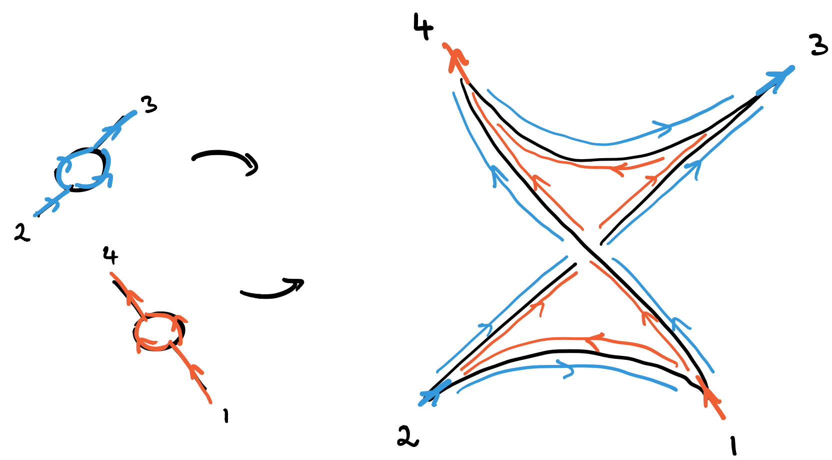
If there was a solution, how would it look like?

$$\underbrace{f_e = 0}_{\hat{q}_e^\pm \,\propto\, p_1^\pm} \quad \text{or} \quad \underbrace{g_e = 0}_{\hat{q}_e^\pm \,\propto\, p_2^\pm} \quad \text{or} \quad \underbrace{f_e = g_e = 0}_{q_e^\pm = 0}$$



Regge limit

But we already know this cannot happen for planar diagrams because momenta in the $\pm$ directions can only flow in one direction along the perimeter:



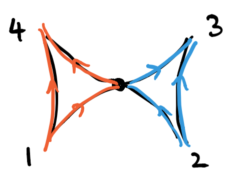
Therefore for the perimeter edges:

$$\mathrm{Im}\,(\hat{q}_e^2 - m_e^2) \;\propto\; \underbrace{f_e}_{\gtrless 0}\; \underbrace{g_e}_{\gtrless 0} \;\neq\; 0$$

definite sign

and the amplitude is analytic.

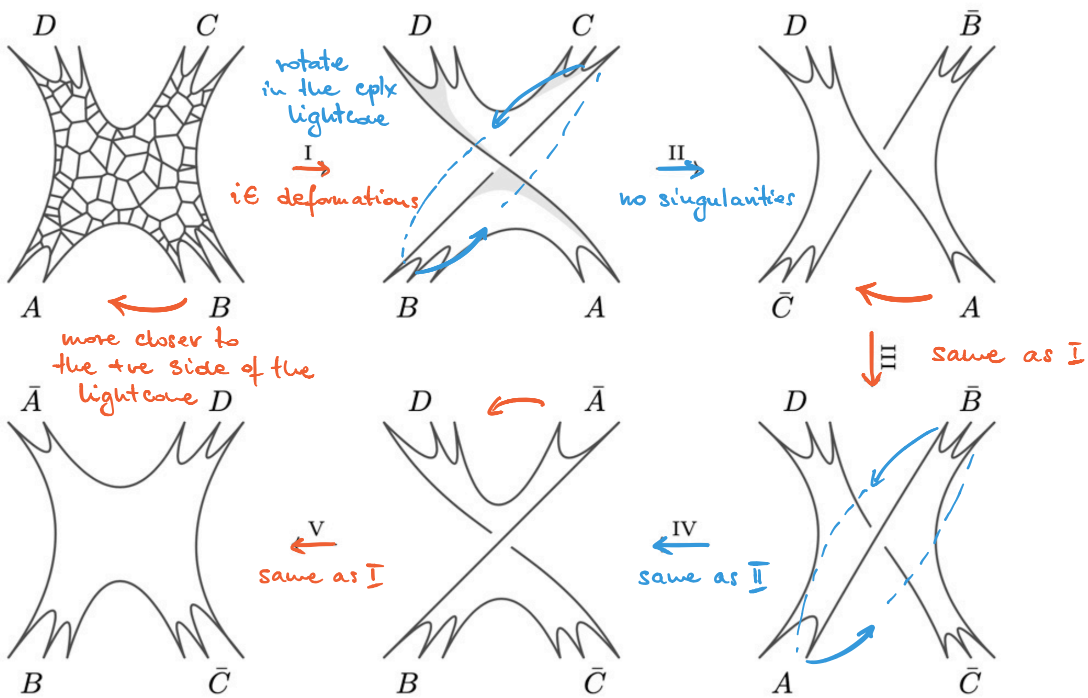
(The only exception are 1- vertex reducible diagrams



which are s- independent anyway)

Putting everything together

The path of analytic continuation for planar diagrams is as follows:



Thanks!