Natural Boundaries for Scattering Amplitudes

Sebastian Mizera (IAS)

based on [hep-th/2210.11448]

What kind of singularities can appear in scattering amplitudes?

- **Poles**
  tree-level Feynman diagrams
  $$\frac{1}{s-m^2}$$

- **Square-root & log branch points**
  loop-level Feynman diagrams
  $$f^{a/2}\log^b f$$

- **Branch points of arbitrary order**
  Regge limit or dimensional regularization
  $$s^{\alpha(t)} \quad \text{or} \quad s^{\epsilon}$$

- **Essential singularities**
  high-energy limit
  $$e^{-s}$$

- **Non-holonomic singularities**
  O(N) sigma models
  $$\frac{1}{1-f^{a/2}\log^b f}$$

- **Accumulation points etc.**

Analytic functions often have another type of singularities: *natural boundaries*

Pole
(real codim-2)

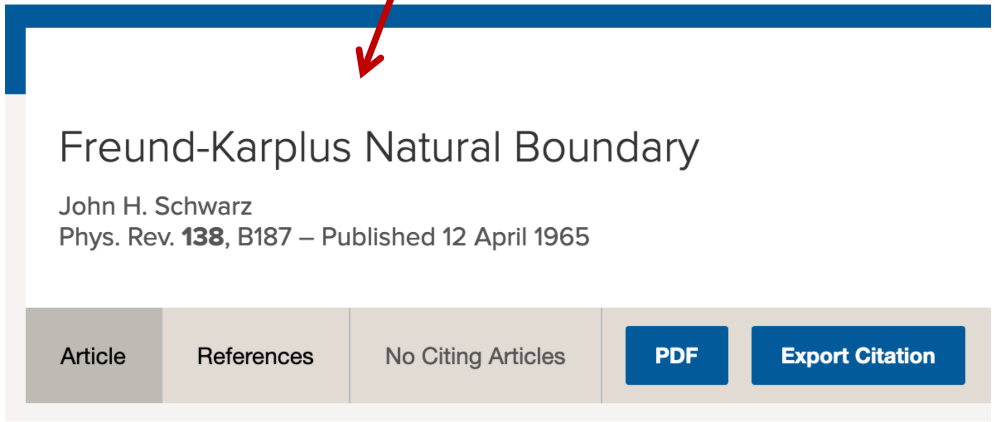
Branch point
(real codim-2)

Natural boundary
(real codim-1)

Can scattering amplitudes have natural boundaries?

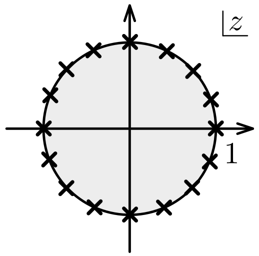
S-matrix archeology

Similar question asked previously by
[Freund, Karplus ‘61], [Schwarz ‘65], [Aks, Gilbert, Howard ‘65]

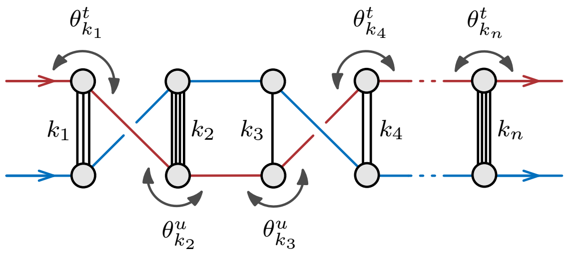


Outline of the talk

1) Toy model for a natural boundary



3) Analytic continuation of unitarity



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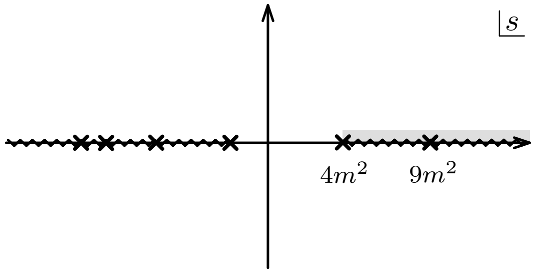
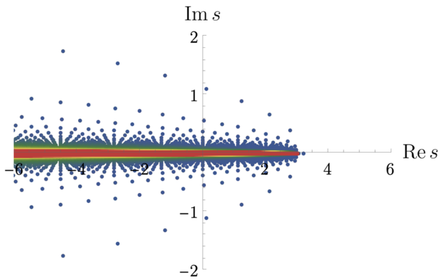


4) Natural boundaries on the second sheet



Lacunary functions: example

$$f(z) = z + z^2 + z^4 + z^8 \underbrace{+}_{\text{Lacuna/gap}} z^{16} + \ldots = \sum_{k=0}^{\infty} z^{2^k}$$

Analytic in the unit disk, $|z| < 1$

Obvious singularity at $z = 1$: $\quad f(1) = 1 + 1 + 1 + \ldots = \infty$

More singularities

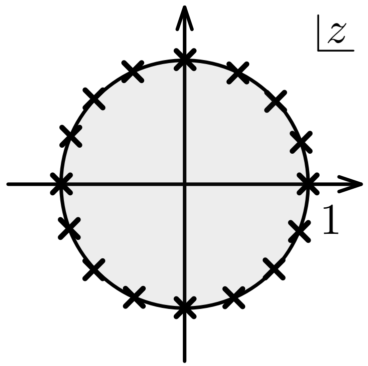
$$\begin{aligned} f(z) &= z + f(z^2) \\ &= z + z^2 + f(z^4) \\ &= z + z^2 + z^4 + f(z^8) \\ &= \ldots . \end{aligned}$$

$\longrightarrow$ $f(-1) = \infty$

$\longrightarrow$ $f(\pm i) = \infty$

$\longrightarrow$ $f(e^{2\pi i \frac{j}{8}}) = \infty$

Singularity at every root of unity of the form $z^{2^k} = 1$



Natural boundary: dense accumulation on the unit circle preventing analytic continuation

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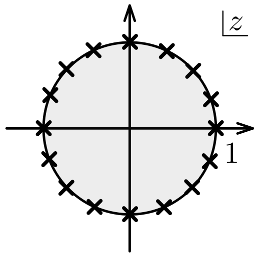

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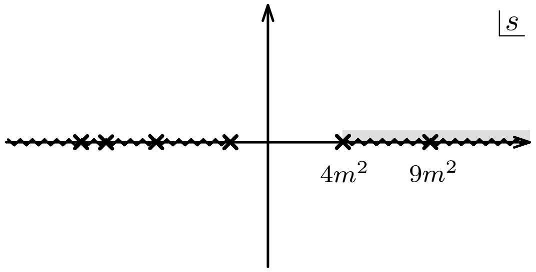

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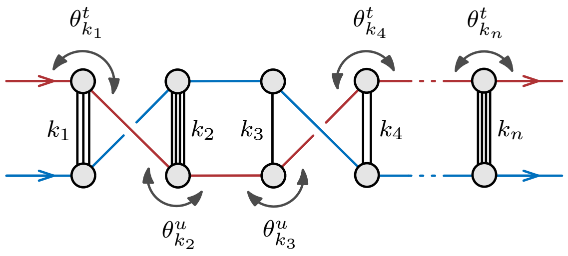

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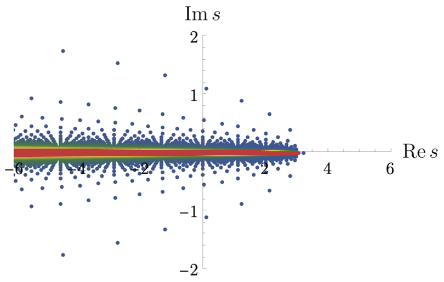
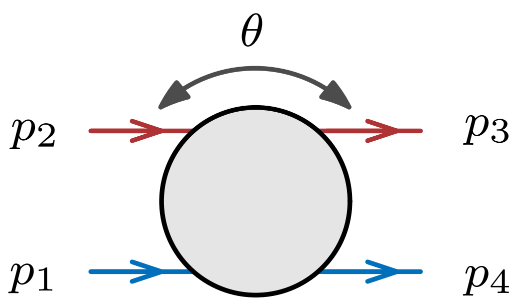


$2 \to 2$ scattering amplitudes of identical scalar particles



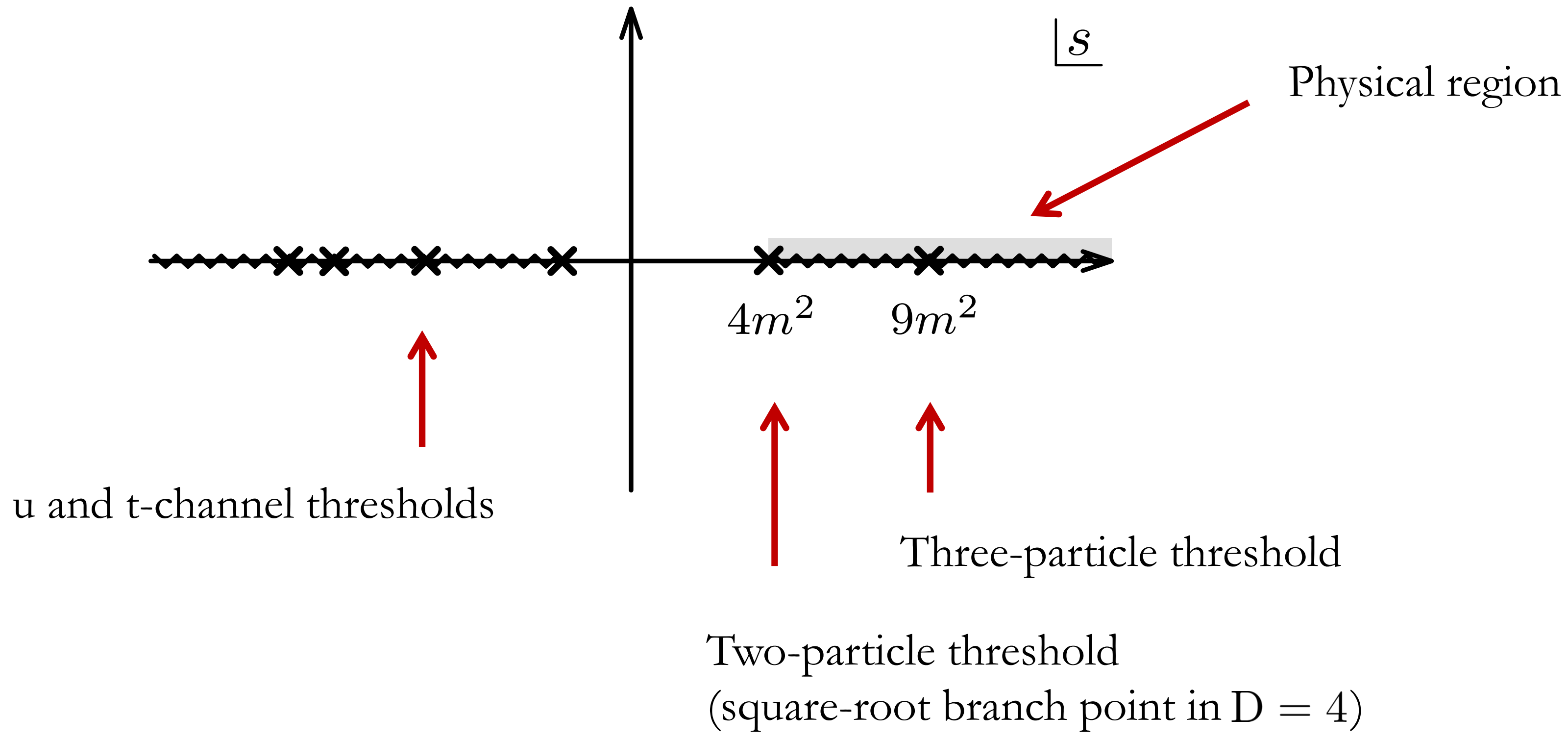
$$= T(s,t) = T(s,z)$$

Mandelstam invariants: $s = (p_1 + p_2)^2, \quad t = (p_2 - p_3)^2, \quad u = (p_1 - p_3)^2$

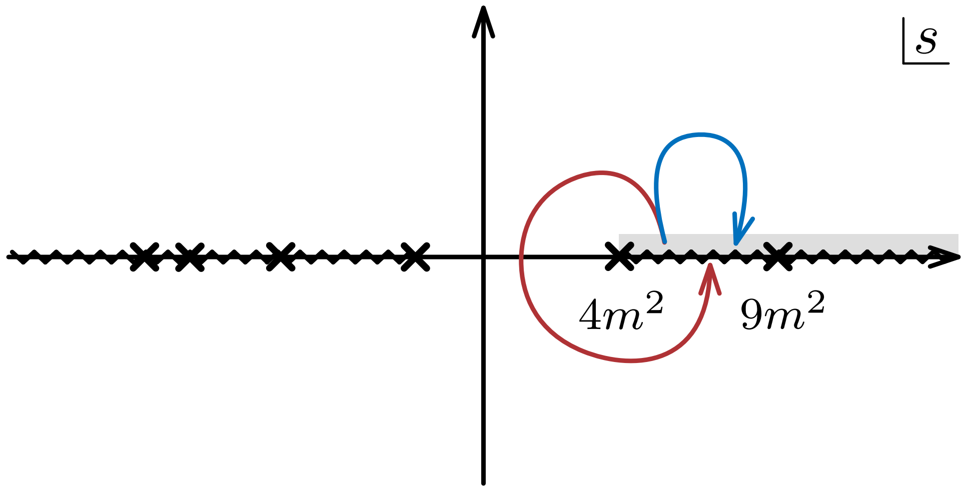
Scattering angle: $z = \cos\theta = 1 + \dfrac{2t}{s - 4m^2}$

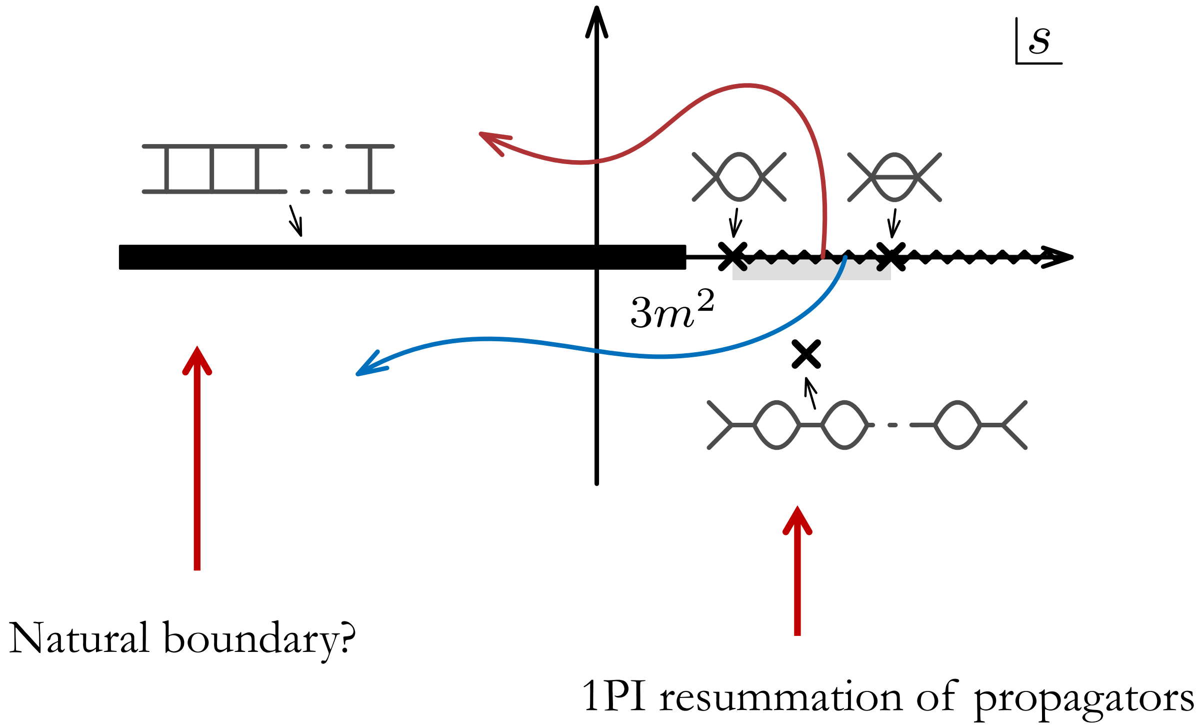
Mass

Complexify s: the physical sheet



Second sheet of the $s > 4m^2$ branch cut





Define as analytic continuations of

Known analytic properties →

$$T_1(s,z) = \lim_{\varepsilon \to 0^+} T(s + i\varepsilon, z)$$

Unknown →

$$T_2(s,z) = \lim_{\varepsilon \to 0^+} T(s - i\varepsilon, z)$$

with

$$4m^2 < s < 9m^2$$

Hence their difference is the analytic continuation of the discontinuity

↑

Use the optical theorem

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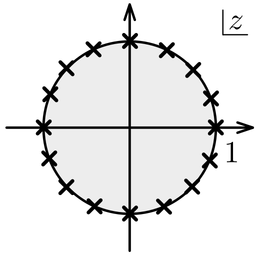

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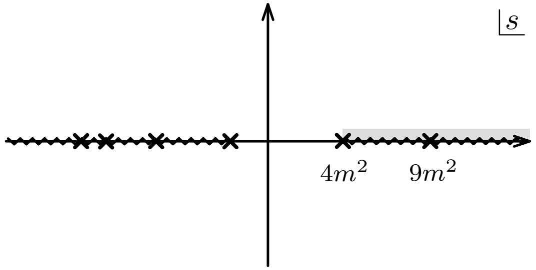

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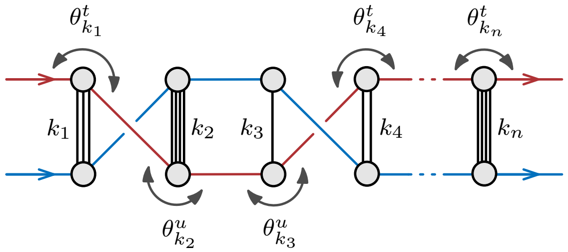

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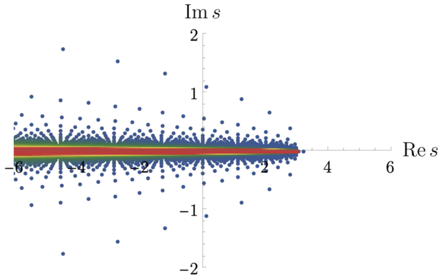
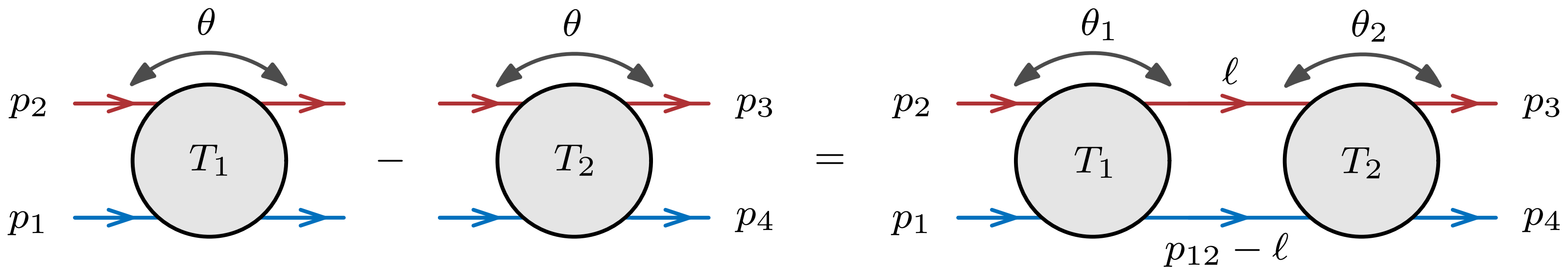


Elastic unitarity

$4m^2 < s < 9m^2$

Unitarity, $SS^\dagger = \mathbb{1}$, embodies the physical principle of probability conservation. Using $S = \mathbb{1} + iT$:

$$\mathrm{Im}\, T = \tfrac{1}{2} T T^\dagger$$



Analytic continuation of elastic unitarity

$$T_1(s,z) - T_2(s,z) = \frac{\sqrt{4m^2 - s}}{\sqrt{s}} \int_{P>0} \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{\sqrt{P(z;z_1,z_2)}}\, T_1(s,z_1)\, T_2(s,z_2),$$

$z_1 = \cos\theta_1$ $\quad$ $z_2 = \cos\theta_2$

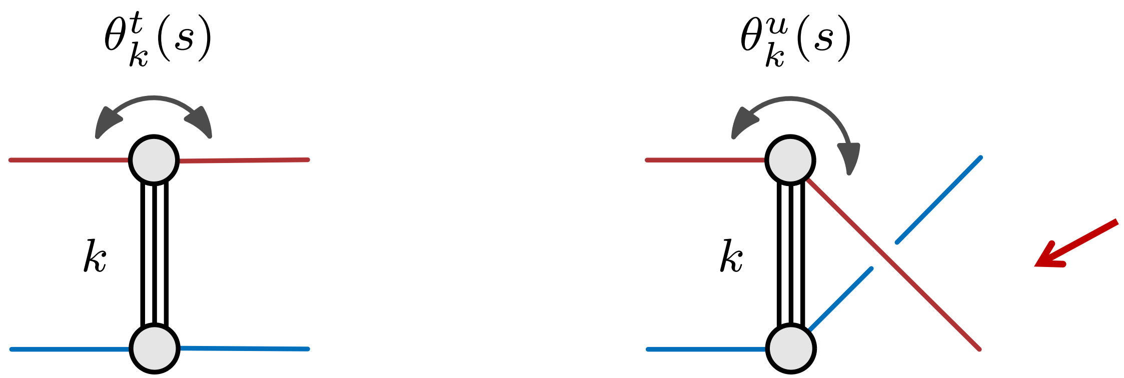
where

$$P(z;z_1,z_2) = 1 - z^2 - z_1^2 - z_2^2 + 2zz_1z_2$$

RHS has a singularity when $P(z^*;z_1^*,z_2^*) = 0 \quad \Leftrightarrow \quad \theta^* = \theta_1^* \pm \theta_2^*$

Discover new singularities iteratively

Both T_1 and T_2 have known t- and u-channel normal thresholds

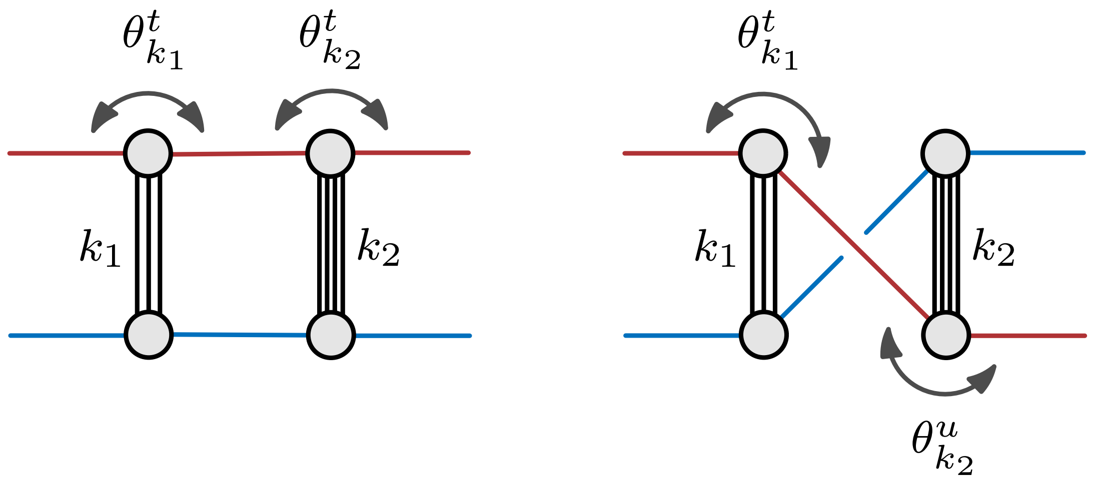


Every edge on-shell
(not Feynman diagrams)

$$\theta_k^{t,u}(s) = \arccos\left[\pm\left(1 + \frac{2k^2m^2}{s-4m^2}\right)\right]$$

Discover new singularities iteratively

Gluing two such singularities gives a new set of (Landau) singularities

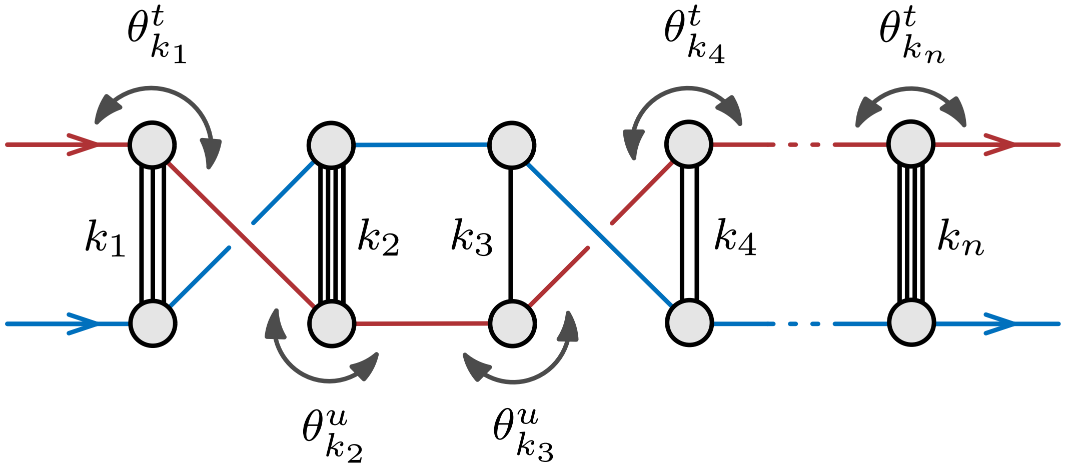


$$\theta^*_{k_1,k_2}(s) = \theta^{t/u}_{k_1}(s) \pm \theta^{t/u}_{k_2}(s)$$

Have to be singularities of T_2

Discover new singularities iteratively

Repeating this over and over gives ladder-type singularities



$$\theta^*_{k_1,k_2,\ldots,k_n}(s) = \sum_{i=1}^{n} \pm\theta^{t,u}_{k_i}(s)$$

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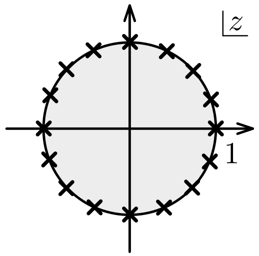

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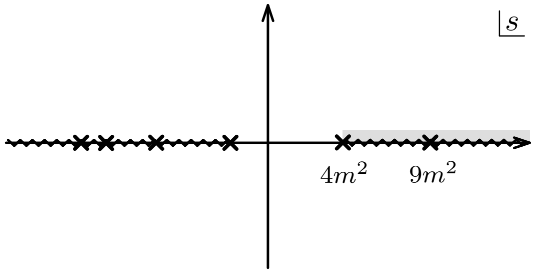

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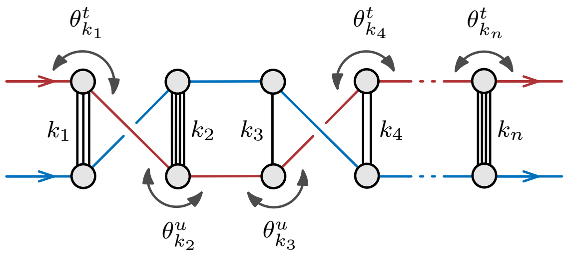

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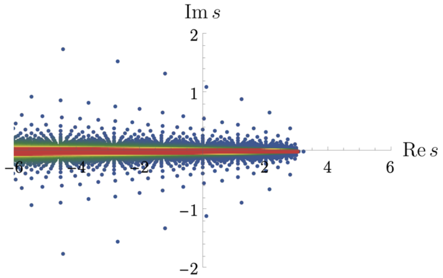


Consider a small subclass of singularities

Planar ladder diagrams with equal number of rungs:

$$\theta^*_{k,k,\ldots,k}(s) = n \arccos\left(1 + \frac{2k^2m^2}{s-4m^2}\right) + 2\pi l$$

n times

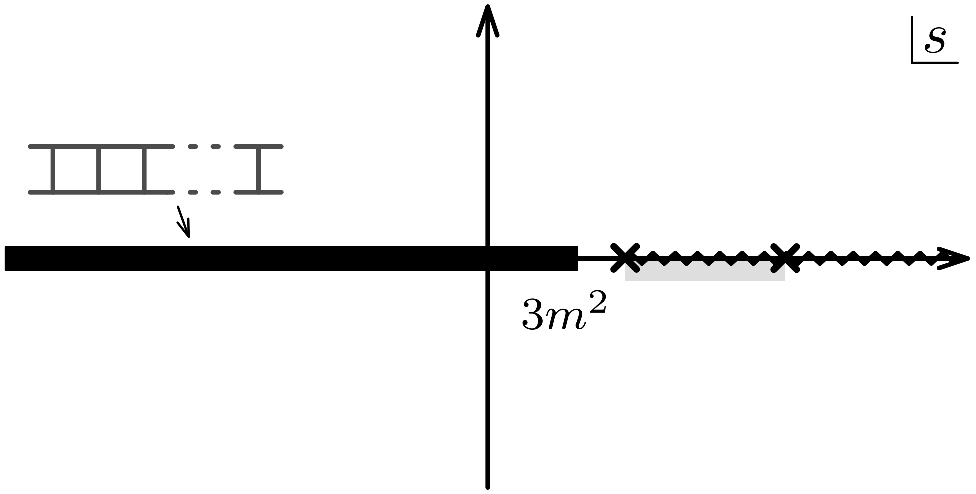
Winding number
$l = 0, 1, \ldots, n-1$

Solving for the singularity in s:

$$s^*_{k,l,n}(\theta) = 2m^2\left(2 + \frac{k^2}{\cos\left(\frac{\theta - 2\pi l}{n}\right) - 1}\right) \leqslant (4-k^2)m^2$$

Natural boundary

Fix the scattering angle θ

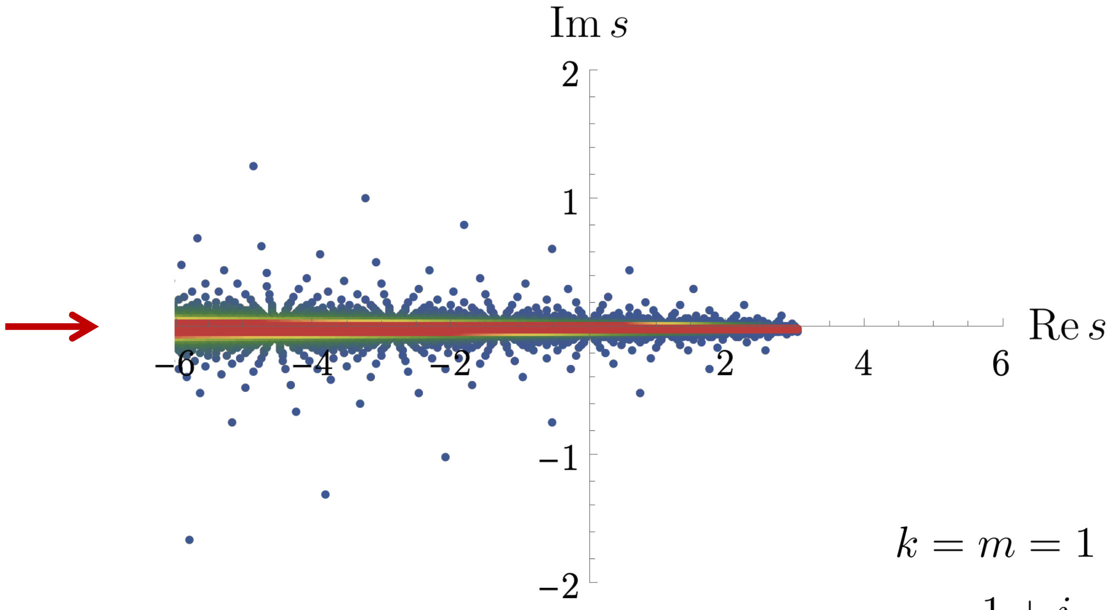


$$s^*_{k,l,n}(\theta) = 2m^2\left(2 + \frac{k^2}{\cos\left(\frac{\theta - 2\pi l}{n}\right) - 1}\right) \leqslant (4 - k^2)m^2$$

Cosine densely covers the interval. $[-1, 1]$ as $n \to \infty$

Natural boundary at fixed complex scattering angle θ ($\mathcal{O}(10^5)$ singularities plotted)

Accumulation as $n \to \infty$

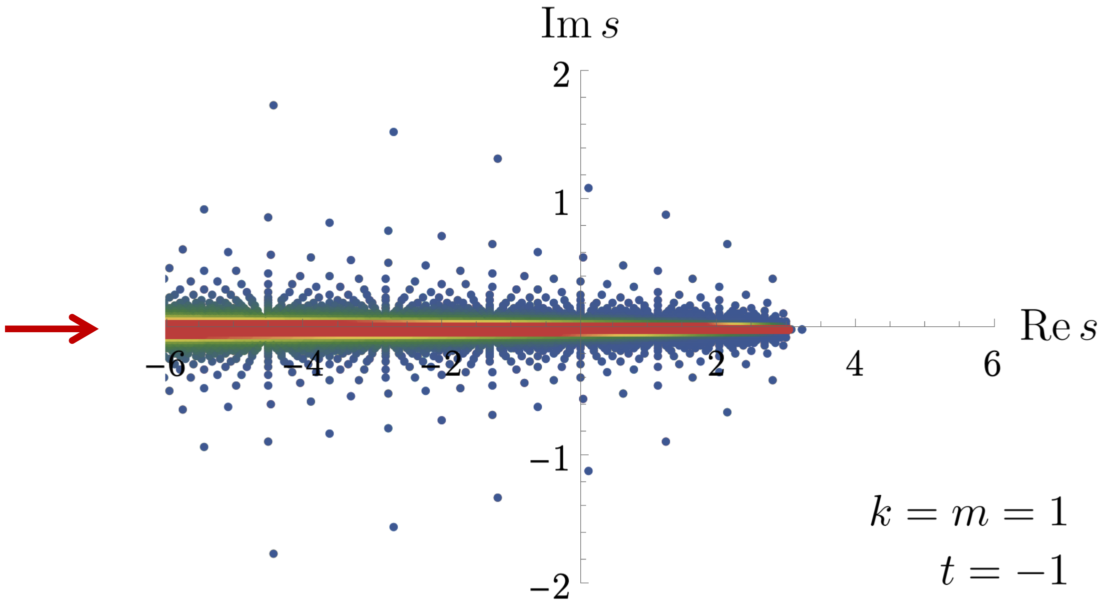


$k = m = 1$

$\theta = \frac{1+i}{2}$

Natural boundary at fixed momentum transfer t ($\mathcal{O}(10^5)$ singularities plotted)

Accumulation as
$n \to \infty$



$k = m = 1$
$t = -1$

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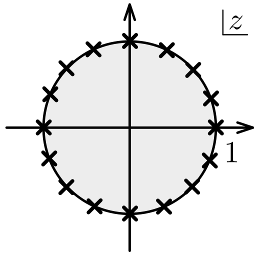

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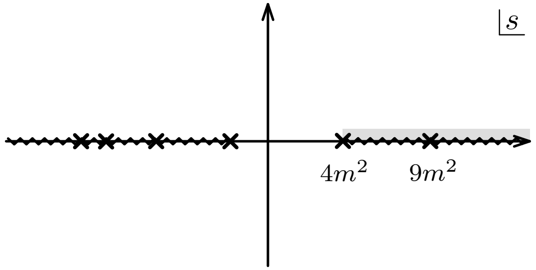

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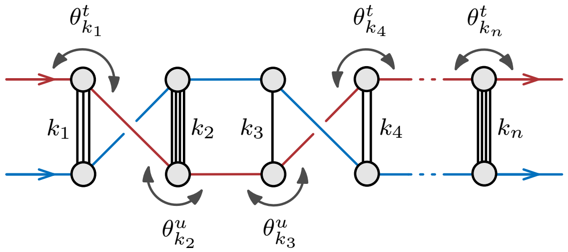

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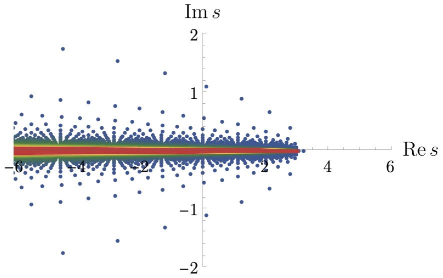

Thank you!