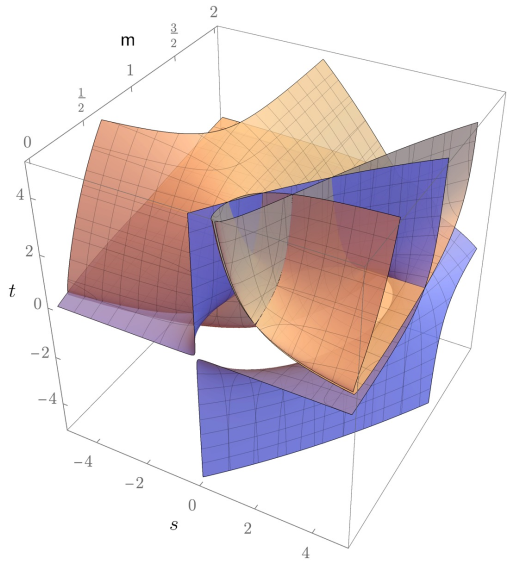
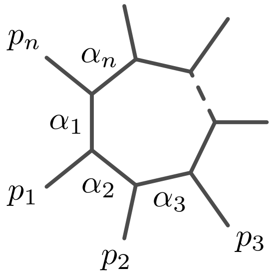


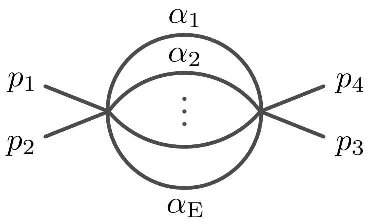
Landau Discriminants

Sebastian Mizera (IAS)

based on **math-ph/2109.08036** with Simon Telen (MPI Leipzig)

We wouldn’t be here
if scattering amplitudes were easy to compute





Divergent Box Integral 15: $I_4^{\{D=4-2\epsilon\}}(m_2^2,p_2^2,p_3^2,m_4^2,p_4^2;s_{12},s_{23};0,m_2^2,0,m_4^2)$

Page contributed by R.K. Ellis

We can calculate this IR divergent box integral from Eq. (2.11) of ref.[1], using the simple replacement rule $\ln\lambda^2 \to \frac{r_\Gamma}{\epsilon} + \ln\mu^2$. We obtain

$$
\begin{aligned}
& I_4^{\{D=4-2\epsilon\}}(m_2^2,p_2^2,p_3^2,m_4^2;t,s;0,m_2^2,0,m_4^2) \\
= \; & \frac{x_s}{m_2 m_4 t(1-x_s^2)} \Bigg\{ \ln x_s \left[-\frac{1}{\epsilon} - \frac{1}{2}\ln x_s - \ln\left(\frac{\mu^2}{m_2 m_4}\right) - \ln\left(\frac{m_2^2-p_2^2}{-t}\right) - \ln\left(\frac{m_4^2-p_3^2}{-t}\right)\right] \\
- \; & \mathrm{Li}_2(1-x_s^2) + \frac{1}{2}\ln^2 y + \sum_{\rho=\pm1} \mathrm{Li}_2(1-x_s y^\rho) \Bigg\}
\end{aligned}
$$

Note the reversal of the arguments s,t to conform with the notation of [1].

$$
y = \frac{m_2\,(m_4^2-p_3^2)}{m_4\,(m_2^2-p_2^2)}
$$

The variable x_s is defined in terms of the function K, such that $x_s = -K(s+i\varepsilon, m_2, m_4)$ and K is given by

$$
\begin{aligned}
K(z,m,m') &= \frac{1-\sqrt{1-4mm'/[z-(m-m')^2]}}{1+\sqrt{1-4mm'/[z-(m-m')^2]}} & z \neq (m-m')^2 \\
K(z,m,m') &= -1 & z = (m-m')^2
\end{aligned}
$$

[QCDloop repository]

$$
I_\ominus(p^2,\underline{\xi}^2) \equiv \frac{i\varpi_r}{\pi}\left(\hat{E}_2\left(\frac{x(P_1)}{x(P_2)}\right) + \hat{E}_2\left(\frac{x(P_2)}{x(P_3)}\right) + \hat{E}_2\left(\frac{x(P_3)}{x(P_1)}\right)\right) \quad \text{mod periods}\,, \tag{142}
$$

where $\hat{E}_2(x)$ is the elliptic dilogarithm

$$
\hat{E}_2(x) = \sum_{n\geq 0}\left(\mathrm{Li}_2(q^n x) - \mathrm{Li}_2(-q^n x)\right) - \sum_{n\geq 1}\left(\mathrm{Li}_2(q^n/x) - \mathrm{Li}_2(-q^n/x)\right). \tag{143}
$$

[review Vanhove '18]

complicated functions of
Mandelstam invariants, masses, etc.
even in the simplest examples

Motivating question:

Can we predict the singularity structure of the S-matrix without explicit computations?

Applications:

- S-matrix bootstrap program
- Dispersion relations & EFT constraints
- Crossing symmetry & imprints of causality on scattering amplitudes

Little bit of progress in systematically answering this question in the 60's
(mostly theories with a mass gap at low-loop orders)

[Bjorken, Landau, Nakanishi, Bros, Epstein, Glaser, Fotiadi, Froissart, Lascoux, Pham, Bogolubov, Steinmann, Ruelle, Araki, Iagolnitzer, Chew, Symanzik, Eden, Landshoff, Oliver, Polkinghorne, Taylor, Cutkovsky, Regge, Chandler, Cahill, Stapp, Wu, Boyling, …]

More recent geometric approaches to massless scattering
(mostly planar $\mathcal{N} = 4$ SYM)

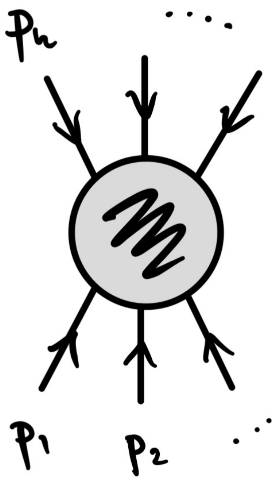
[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka, Dennen, Spradlin, Volovich, Prlina, Stankowicz, Stanojevic, Gürdoğan, Parisi, …]

Plan for the talk:

- What is the physical meaning of singularities?
  – Saddle points in the wordline formalism
  – Fluctuations around saddles

- Applying tools from computational algebraic geometry
  – Landau discriminants
  – Results and the package `Landau.jl`

Path integral for a given scattering process:



$$= \sum_{\substack{\text{worldline} \\ \text{topologies}}} \int \mathcal{D}X \, e^{\frac{i}{\hbar} S_{\mathrm{WL}}[X, p_i]}$$

worldline action (free)

Feynman diagrams

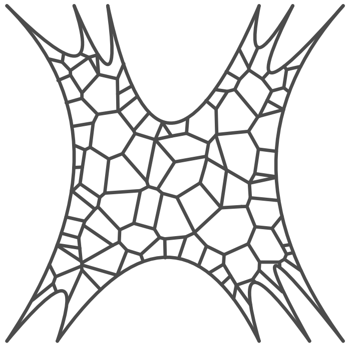
external momenta

Contribution from a single worldline Feynman diagram:

(equivalent to loop momentum integration)

Schwinger parameters $\alpha_{e=1,2,\ldots,\mathrm{E}}$

(localized) worldline action



$$= \int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}} \, \mathrm{N} \, e^{\frac{i}{\hbar}\mathcal{V}}$$

polynomial numerator encoding interactions

determinant of the Laplacian

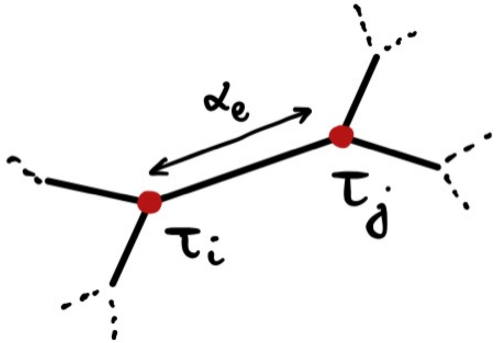
The worldline action takes the form

$$\mathcal{V} = -\sum_{i<j} p_i \cdot p_j \, \mathcal{G}_{ij} - \sum_e m_e^2 \, \alpha_e$$

Green's function

internal masses

Green's functions treated as functions of Schwinger parameters α_e



Non-trivial fact:

All kinematic singularities arise in the classical limit, $\hbar \to 0$

Therefore, they can be determined by saddle-point equations

$$\underbrace{\alpha_e}_{\text{boundary saddles}} \underbrace{\frac{\partial \mathcal{V}}{\partial \alpha_e}}_{\text{bulk saddles}} = 0, \qquad e = 1, 2, \ldots, \mathrm{E}$$

boundary saddles bulk saddles (putting the e^{th} edge on-shell)

Wordline action is special because of homogeneity under dilations:

$$\mathcal{V}(\lambda\alpha_e) = \lambda\mathcal{V}(\alpha_e)$$

Three important consequences:

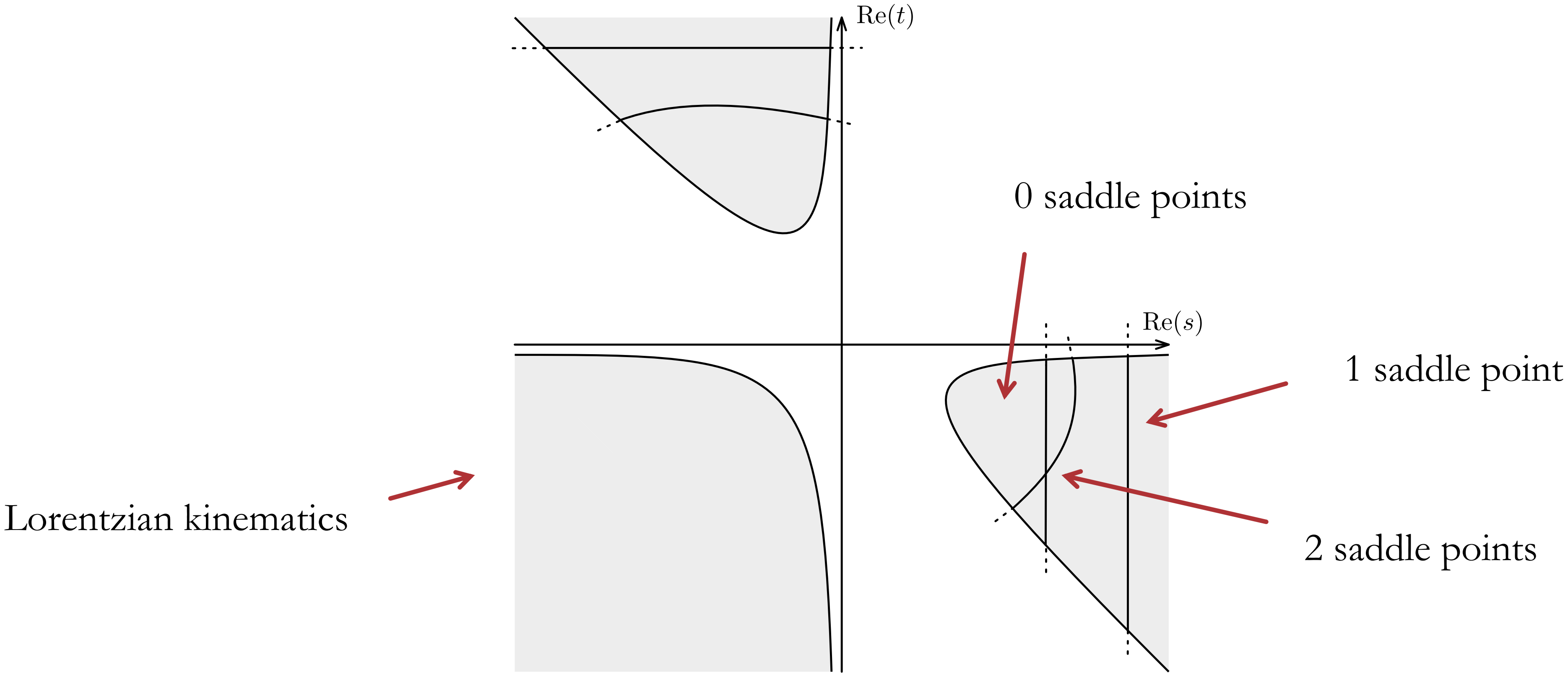
- The action vanishes on the saddle points, $\mathcal{V} = \sum_{e=1}^{\mathrm{E}} \alpha_e \frac{\partial \mathcal{V}}{\partial \alpha_e} = 0$

- Integrating out the overall scale gives $\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1-d}} e^{\frac{i}{\hbar}\lambda\mathcal{V}} \propto \frac{1}{\mathcal{V}^d}$

degree of divergence

- E equations on E−1 independent Schwinger parameters, leaving at least one constraint on the *external kinematics*

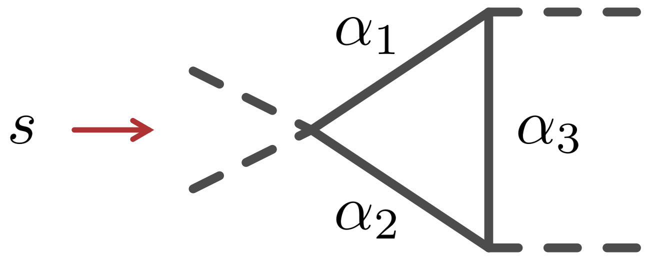
For instance at 4-pt, in terms of the Mandelstam invariants s and t



These are known as *anomalous* (or normal) *thresholds*:
intrinsically *Lorentzian* phenomena,
at least partially encoding causality in perturbation theory

- Most singularities of the S-matrix have complex Schwinger parameters & Mandelstam invariants
- Kinematic space has an extremely complicated sheet structure

Elementary example:



internal masses m

external masses M

$$\mathcal{V} = \frac{s\alpha_1\alpha_2 + M^2(\alpha_1{+}\alpha_2)\alpha_3}{\alpha_1+\alpha_2+\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3)$$

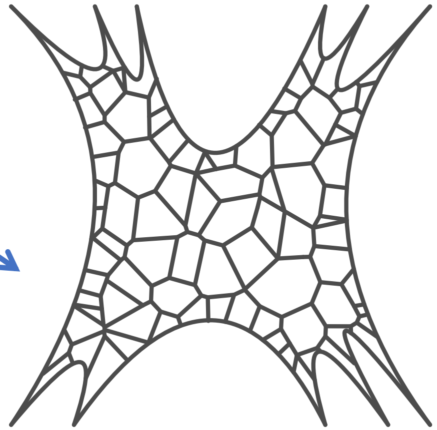
Bulk saddle: $s = M^2(4m^2 - M^2)/m^2$

Boundary saddle at $\alpha_3 = 0$: $s = 4m^2$

To summarize, the perturbative S-matrix has singularities of the form:

worldline formalism

Schwinger parameter
$\alpha_e \in \mathbb{C}$
(proper time)

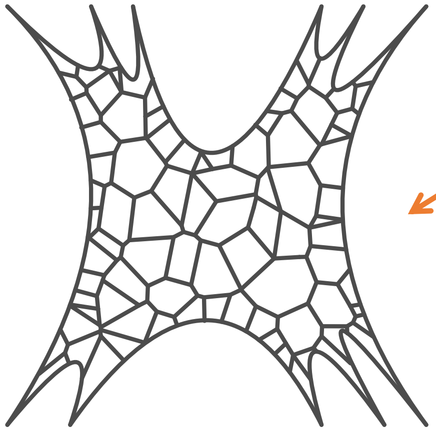


saddle points of the worldline action $\mathcal{V}$

$$\frac{\partial \mathcal{V}}{\partial \alpha_e} = 0$$

loop momentum space

internal momentum q_e^μ

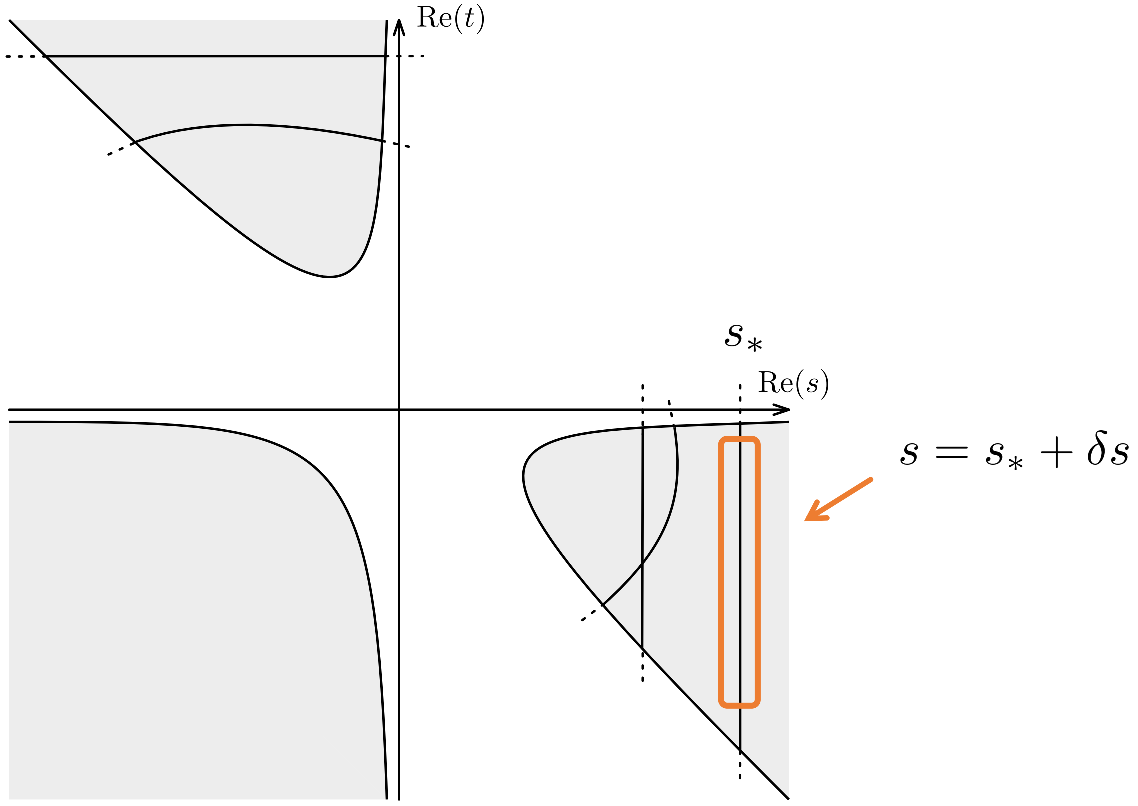


internal momenta going on-shell

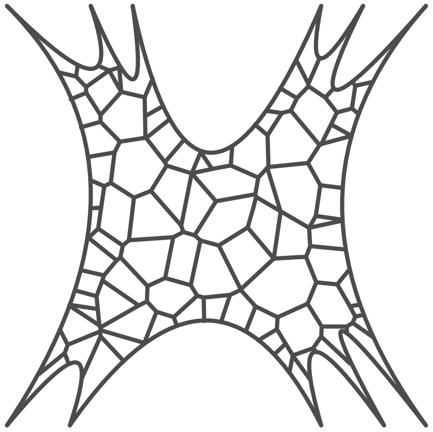
$$q_e^2 - m_e^2 = 0$$

(plus additional constraints)

The type of singularity is determined by studying fluctuations around the saddle points



If the critical point is isolated and non-degenerate:



$$\approx \# \frac{\mathrm{N}}{\mathcal{U}^{\mathrm{D}/2}(\det' \mathbf{H})^{1/2}} \begin{cases} (\delta s)^{\gamma} \\ (\delta s)^{\gamma} \log(\delta s) \end{cases}$$

localizing the path integral

localizing the Schwinger parameters

locality implies $-1 \leqslant \gamma \in \frac{\mathbb{Z}}{2}$

[with Hannesdottir]

[see also Polkinghorne, Screaton '60]

Solving for saddle points becomes an extremely complicated task and – until recently – only a handful of examples were known

Simplification:

The action is always a ratio

$$\mathcal{V} = \frac{\mathcal{F}}{\mathcal{U}},$$

where the two *Symanzik polynomials* are given by

$$\mathcal{U} = \sum_{\substack{\text{spanning}\\ \text{trees } T}} \prod_{e \notin T} \alpha_e, \qquad \mathcal{F} = \sum_{\substack{\text{spanning}\\ \text{2-trees } T_L \sqcup T_R}} p_L^2 \prod_{e \notin T_L, T_R} \alpha_e - \mathcal{U} \sum_{e=1}^{\mathrm{E}} m_e^2 \, \alpha_e$$

Therefore singularities are determined by a polynomial system of equations

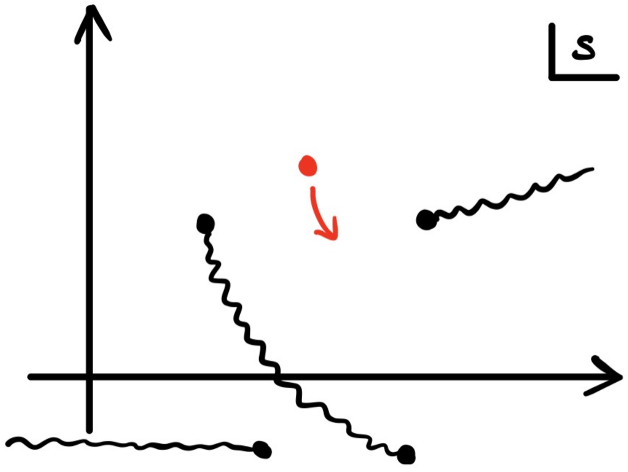
$$\frac{\partial \mathcal{V}}{\partial \alpha_e} = 0 \qquad \Leftrightarrow \qquad \frac{\partial \mathcal{F}}{\partial \alpha_e} = 0, \qquad \alpha_1 \alpha_2 \cdots \alpha_{\mathrm{E}} \, \mathcal{U} \neq 0$$

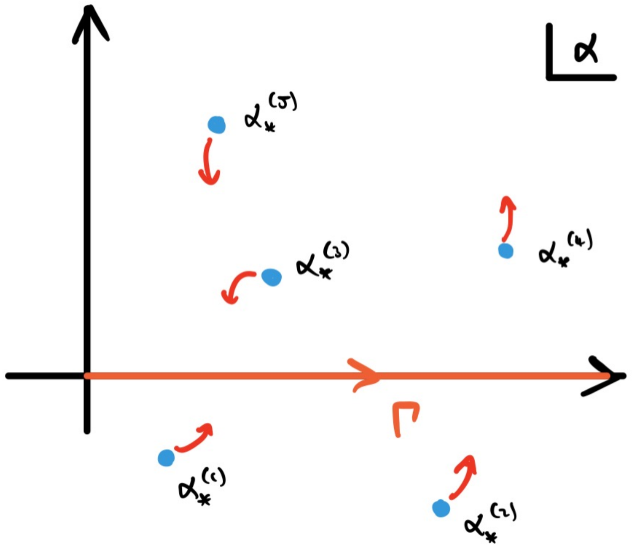
modding out by *subleading* and *second-type* singularities

[Bjorken, Landau, Nakanishi '59]

Toy model in a single variable, $(1 : \alpha) \in \mathbb{CP}^1$:

$$\int_\Gamma e^{\frac{i}{\hbar}\mathcal{F}} \qquad \text{with} \qquad \mathcal{F}(\alpha, s) = \prod_i (\alpha - \alpha_*^{(i)})$$



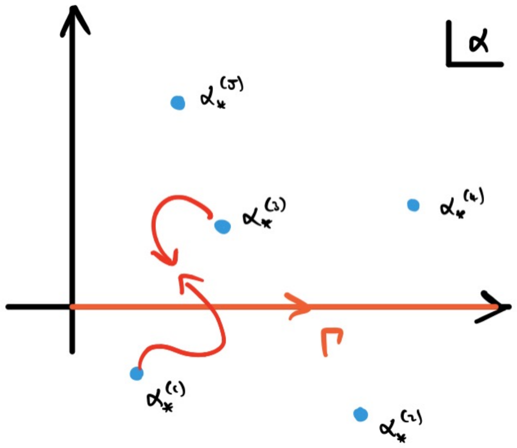


Singularity can only occur when at least two roots coincide:

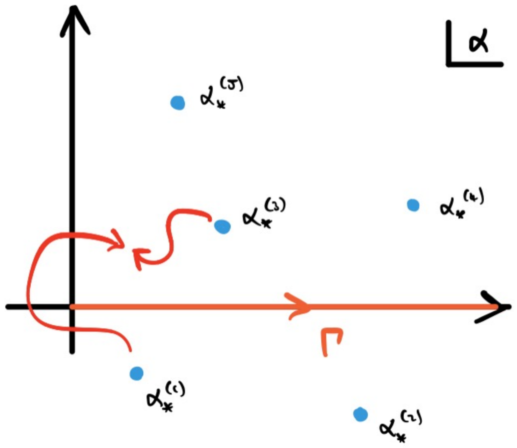
$$\Delta = \prod_{i \neq j} (\alpha_*^{(i)} - \alpha_*^{(j)}) = 0$$

discriminant

Only a *necessary* condition for a singularity:

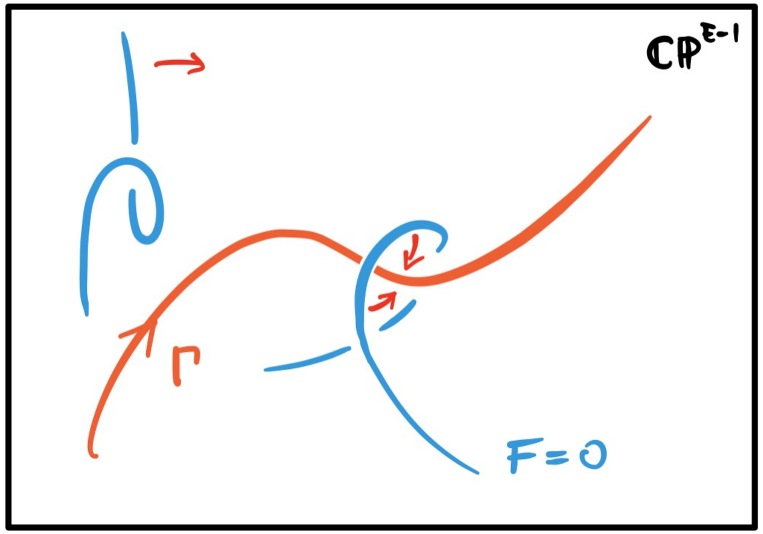


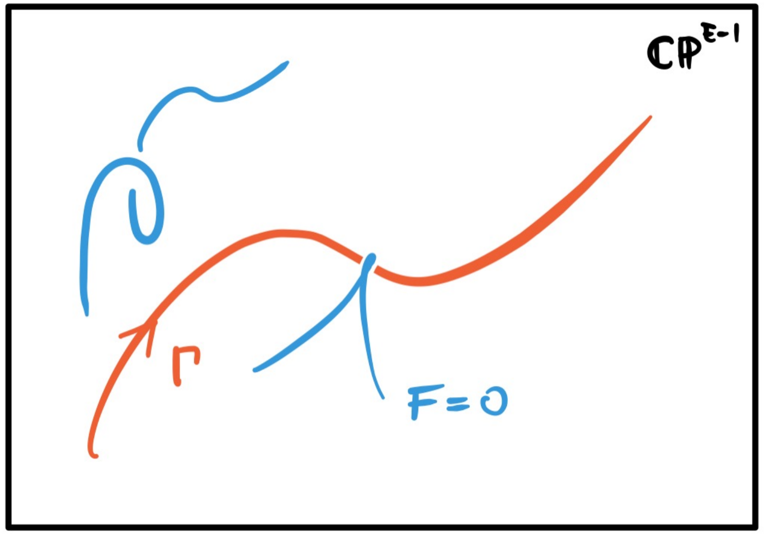
singular



non-singular

Generalization to higher dimensions:





$\mathcal{F}(\alpha_e, s_I) = 0$ degenerates $\Rightarrow$ Landau discriminant $\Delta(s_I) = 0$

Landau discriminant gives the singularity locus of Feynman integrals

(named after [Landau '59])

For the mathematicians in the audience, let us be more precise:

$$X := \mathbb{P}^{\mathrm{E}_G - 1} \setminus V_{\mathbb{P}^{\mathrm{E}_G - 1}}(\alpha_1 \alpha_2 \cdots \alpha_{\mathrm{E}_G} \mathcal{U}_G)$$

space of Schwinger parameters

$$Y := \left\{ (\alpha, q) \in X \times \mathbb{P}(\mathcal{K}_G) \;\middle|\; \frac{\partial \mathcal{F}_G}{\partial \alpha_e}(\alpha; q) = 0,\ \ e = 1, 2, \ldots, \mathrm{E}_G \right\}$$

incidence variety

$$\pi_{\mathbb{P}(\mathcal{K}_G)} : \ Y \to \mathbb{P}(\mathcal{K}_G)$$

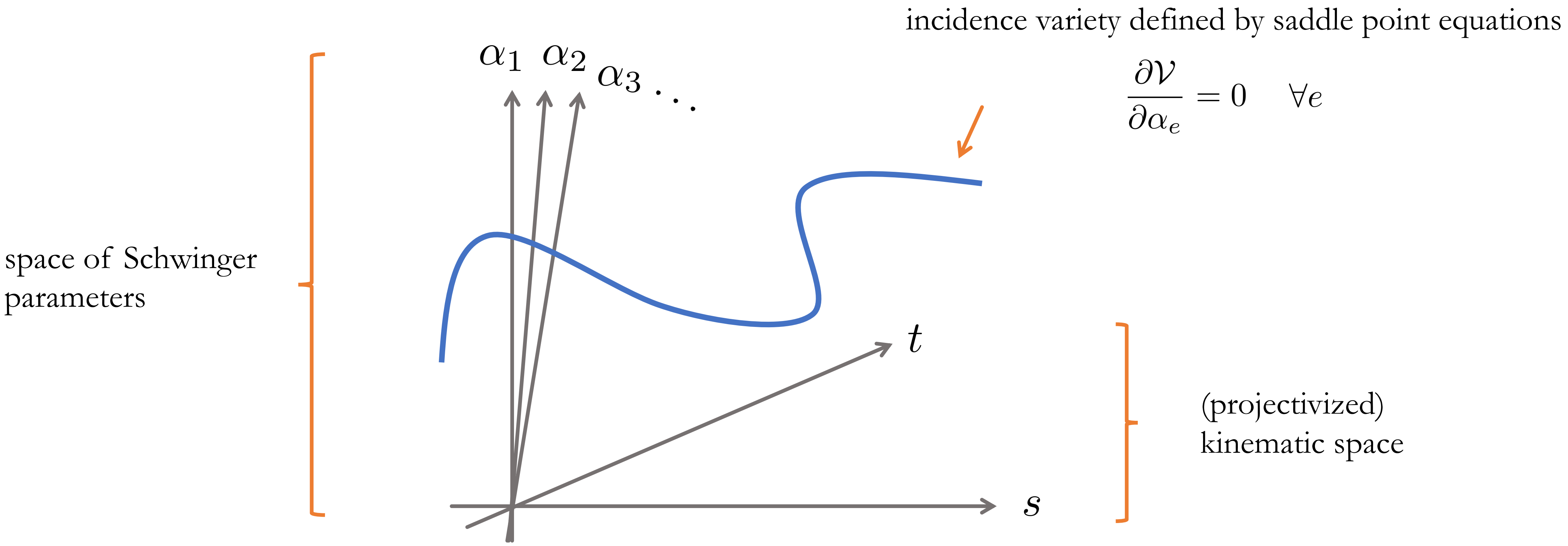
projection to the projectivized kinematic space

Definition 4 (Landau discriminant) The *Landau discriminant* ∇_G of a Feynman diagram G is the subvariety of $\mathbb{P}(\mathcal{K}_G)$ given by the Zariski closure

$$\nabla_G := \overline{\pi_{\mathbb{P}(\mathcal{K}_G)}(Y)} \subset \mathbb{P}(\mathcal{K}_G) \tag{12}$$

of $\pi_{\mathbb{P}(\mathcal{K}_G)}(Y)$ in $\mathbb{P}(\mathcal{K}_G)$. If ∇_G is a hypersurface, its defining polynomial Δ_G ($\nabla_G =: \{\Delta_G = 0\}$), which is unique up to scaling, is called the *Landau discriminant polynomial*. If ∇_G has codimension greater than 1, we set $\Delta_G = 1$.

Def. 4 in [math-ph/2109.08036 with Telen]


incidence variety defined by saddle point equations
$\frac{\partial \mathcal{V}}{\partial \alpha_e} = 0 \quad \forall e$
α_1 α_2 α_3 ...
space of Schwinger parameters
t
s
(projectivized) kinematic space

Some concrete questions about Landau discriminants:
For a given Feynman diagram

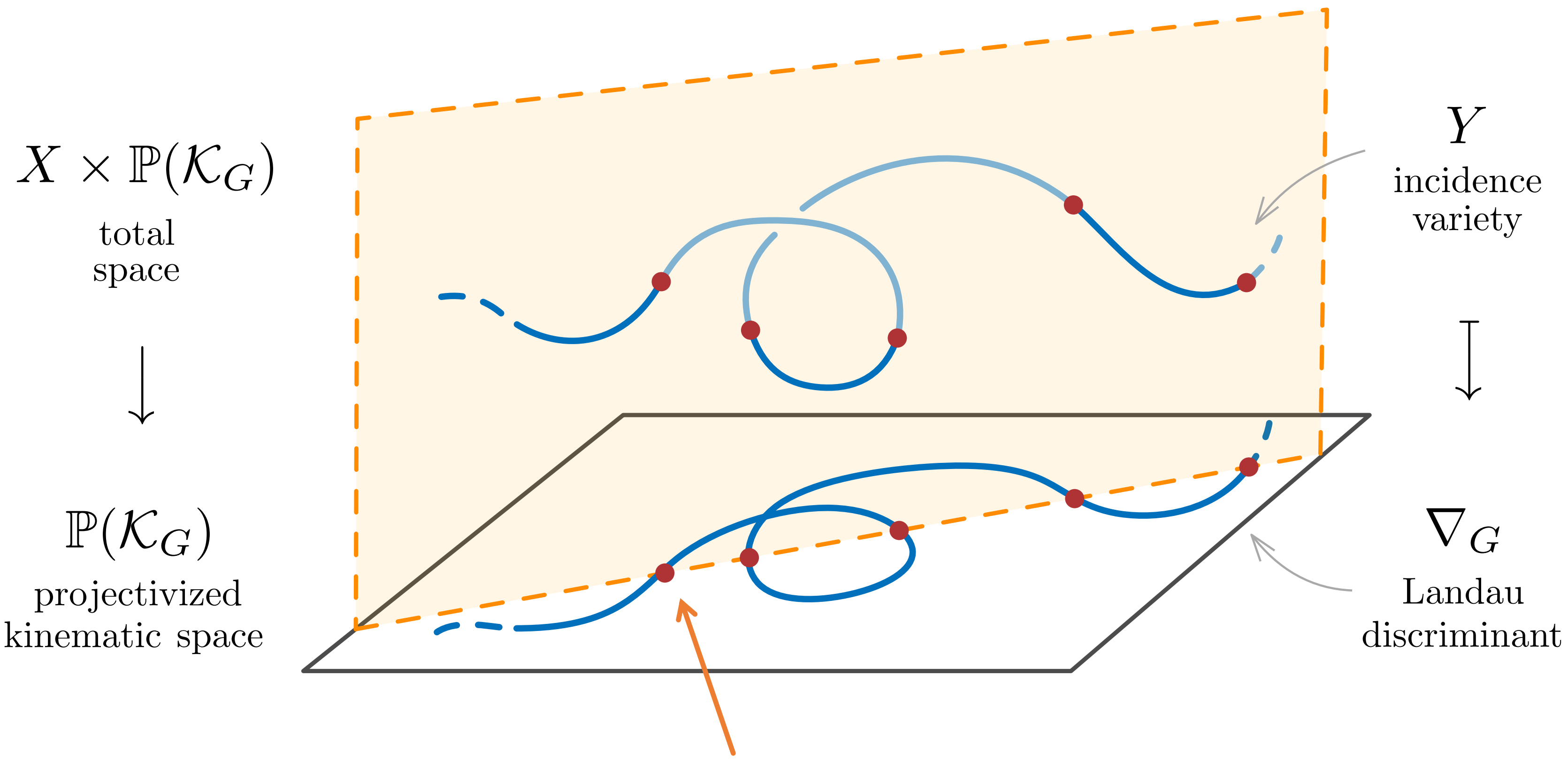
- what's the dimension?
- what's the degree?
- how to compute it?

For distinct internal masses one can show that the Landau discriminant is of *at least* codimension-1 in the kinematic space

Thm. 1 in [math-ph/2109.08036 with Telen]

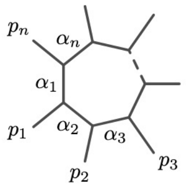
(This is what distinguishes threshold singularities from UV/IR divergences)

To get the degree, simply intersect with a generic line in the total space

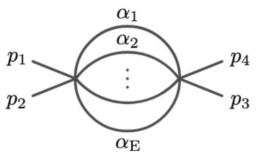


red points = degree (how complicated the singularity is)

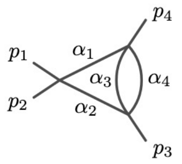
Some simple examples I



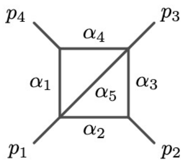
(a) One-loop n-gon diagram, $G = \mathtt{A}_n$ (Sec. 2.5)



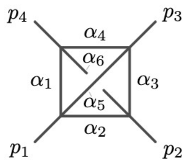
(b) Banana diagram with E edges, $G = \mathtt{B}_E$ (Sec. 2.6, 4.4)



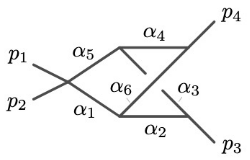
(c) Parachute diagram, $G =$ `par` (Ex. 15, Thm. 2)



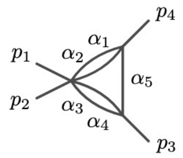
(d) Acnode diagram, $G =$ `acn` (Ex. 10, Rk. 9, Thm. 2)



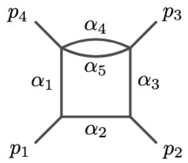
(e) Envelope diagram, $G =$ `env` (Ex. 12, Sec. 3.4)



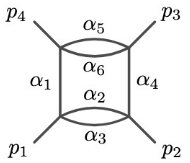
(f) Non-planar triangle-box diagram, $G =$ `npltrb` (Thm. 2)



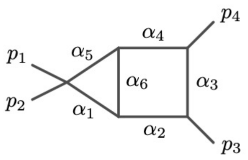
(g) Twice doubled-edge triangle diagram, $G =$ `tdetri` (Thm. 2)



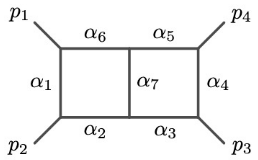
(h) Doubled-edge box diagram, $G =$ `debox` (Thm. 2)



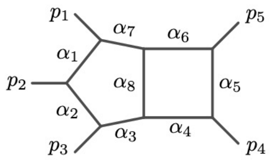
(i) Twice doubled-edge box diagram, $G =$ `tdebox` (Thm. 2)



(j) Planar triangle-box diagram, $G =$ `pltrb` (Sec. 3.3.1, Thm. 2)



(k) Double-box diagram, $G =$ `dbox` (Thm. 2)



(l) Penta-box diagram, $G =$ `pentb` (Ex. 13)

Diagram G	codim ∇_G	deg ∇_G	time (sec)
`par`	1	6	0.176
`acn`	1	16	0.489
`env`	1	114	13.1
`npltrb`	2	10	37.2
`tdetri`	1	12	1.04
`debox`	1	8	0.366
`tdebox`	1	16	10.5
`pltrb`	2	9	24.3
`dbox`	1	12	8.64
`pentb`	1	14	62.8

not necessarily codimension-1

Tab. 1 in [math-ph/2109.08036 with Telen]

Strategy for computing the Landau discriminant:

- Write the (finite) ansatz, e.g., $\Delta = \sum_{i,j,k,l} c_{ijkl} s^i t^j \mathsf{M}^k \mathsf{m}^l = 0$

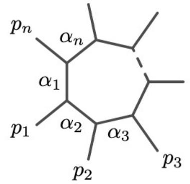
- Keep intersecting with random lines until all c_{ijkl}'s are determined

- Use *homotopy continuation* to recycle the results

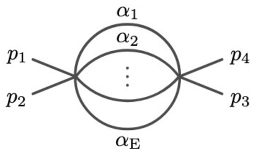
You can try it at home with the Julia package `Landau.jl`!

https://mathrepo.mis.mpg.de/Landau/

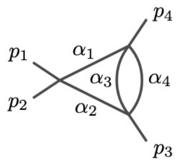
Some simple examples II



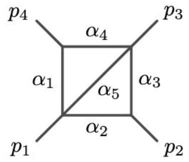
(a) One-loop n-gon diagram, $G = \mathtt{A}_n$ (Sec. 2.5)



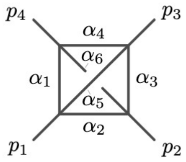
(b) Banana diagram with E edges, $G = \mathtt{B}_E$ (Sec. 2.6, 4.4)



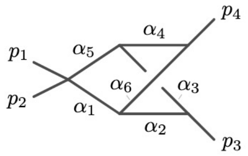
(c) Parachute diagram, $G = \mathtt{par}$ (Ex. 15, Thm. 2)



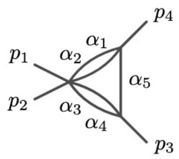
(d) Acnode diagram, $G = \mathtt{acn}$ (Ex. 10, Rk. 9, Thm. 2)



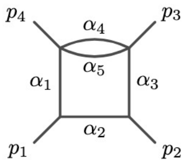
(e) Envelope diagram, $G = \mathtt{env}$ (Ex. 12, Sec. 3.4)



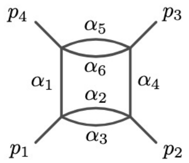
(f) Non-planar triangle-box diagram, $G = \mathtt{npltrb}$ (Thm. 2)



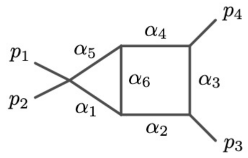
(g) Twice doubled-edge triangle diagram, $G = \mathtt{tdetri}$ (Thm. 2)



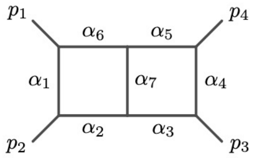
(h) Doubled-edge box diagram, $G = \mathtt{debox}$ (Thm. 2)



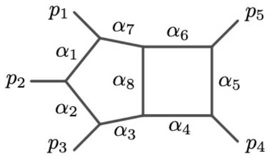
(i) Twice doubled-edge box diagram, $G = \mathtt{tdebox}$ (Thm. 2)



(j) Planar triangle-box diagram, $G = \mathtt{pltrb}$ (Sec. 3.3.1, Thm. 2)



(k) Double-box diagram, $G = \mathtt{dbox}$ (Thm. 2)



(l) Penta-box diagram, $G = \mathtt{pentb}$ (Ex. 13)

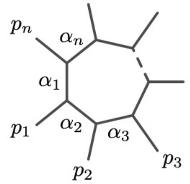
Diagram G	$\nabla_G(\mathcal{E})$	$t_{\mathtt{symb}}$	$t_{\mathtt{num}}$
`par`	$[1,2]_1$	0.2	0.5
`acn`	$[4,8]_1$	175306.0	1.4
`env`	$[8,8,8,9,12]_1$	×	1226.1
`npltrb`	$[1,1]_1$	1.9	4.0
`tdetri`	$[2]_1,[1]_2$	8.1	1.2
`debox`	$[3]_1,[1]_2$	7.9	0.5
`tdebox`	$[2]_1,[1]_2$	1476.8	4.3
`pltrb`	$[1,1]_2$	0.6	×
`dbox`	$[2,4]_1$	13634.2	4.5
`pentb`	$[12]_1$	×	815.9

classic elimination theory with `Macaulay2`

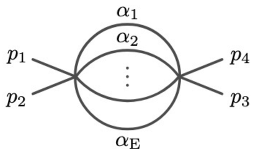
`Landau.jl`

Tab. 1 in [math-ph/2109.08036 with Telen]

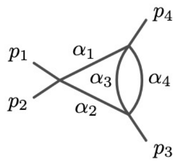
Some simple examples III



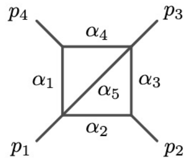
(a) One-loop n-gon diagram, $G = \texttt{A}_n$ (Sec. 2.5)



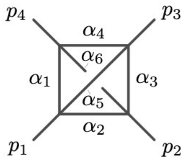
(b) Banana diagram with E edges, $G = \texttt{B}_{\mathrm{E}}$ (Sec. 2.6, 4.4)



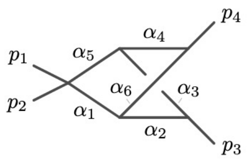
(c) Parachute diagram, $G = \texttt{par}$ (Ex. 15, Thm. 2)



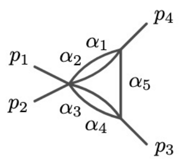
(d) Acnode diagram, $G = \texttt{acn}$ (Ex. 10, Rk. 9, Thm. 2)



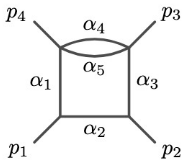
(e) Envelope diagram, $G = \texttt{env}$ (Ex. 12, Sec. 3.4)



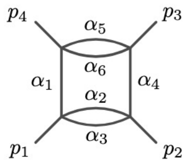
(f) Non-planar triangle-box diagram, $G = \texttt{npltrb}$ (Thm. 2)



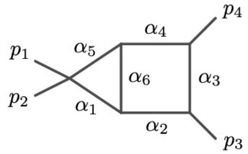
(g) Twice doubled-edge triangle diagram, $G = \texttt{tdetri}$ (Thm. 2)



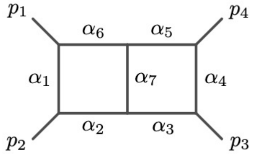
(h) Doubled-edge box diagram, $G = \texttt{debox}$ (Thm. 2)



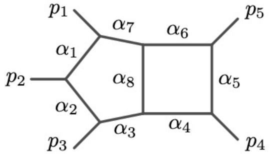
(i) Twice doubled-edge box diagram, $G = \texttt{tdebox}$ (Thm. 2)



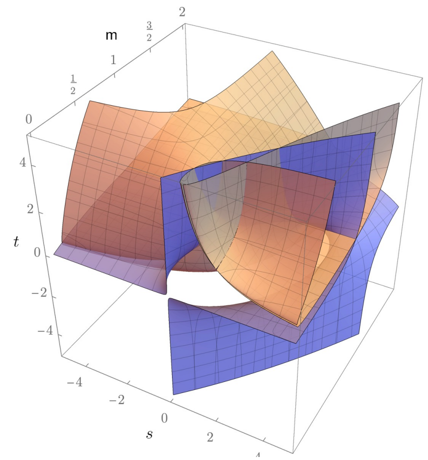
(j) Planar triangle-box diagram, $G = \texttt{pltrb}$ (Sec. 3.3.1, Thm. 2)



(k) Double-box diagram, $G = \texttt{dbox}$ (Thm. 2)



(l) Penta-box diagram, $G = \texttt{pentb}$ (Ex. 13)

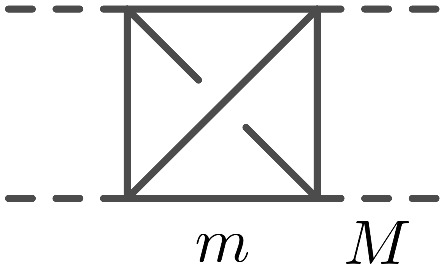


$$\nabla_{\texttt{par}}(\mathcal{E}) = \{(\mathsf{M}-\mathsf{m})(\mathsf{M}^2 - 10\mathsf{Mm} + 9\mathsf{m}^2 + 4\mathsf{m}s) = 0\},$$
$$\nabla_{\texttt{npltrb}}(\mathcal{E}) = \{\mathsf{m}(s-\mathsf{M}) = 0\},$$
$$\nabla_{\texttt{tdetri}}(\mathcal{E}) = \{s - 4\mathsf{M} = s - 4\mathsf{m} = 0\} \cup \{9\mathsf{m}^2 - 10\mathsf{mM} + \mathsf{m}s + \mathsf{M}^2 = 0\},$$
$$\nabla_{\texttt{debox}}(\mathcal{E}) = \{t - \mathsf{M} = t - \mathsf{m} = 0\} \cup \{36\mathsf{m}^2\mathsf{M} - 9\mathsf{m}^2 s - 28\mathsf{mM}t + 10\mathsf{m}st + 4\mathsf{m}t^2 + 4\mathsf{M}^2 t - st^2 = 0\},$$
$$\nabla_{\texttt{tdebox}}(\mathcal{E}) = \{t - 4\mathsf{m} = 40\mathsf{m} + 4\mathsf{M} - 11t = 0\} \cup \{36\mathsf{m}^2 - 40\mathsf{mM} + 16\mathsf{m}s + 4\mathsf{m}t + 4\mathsf{M}^2 - st = 0\},$$
$$\nabla_{\texttt{pltrb}}(\mathcal{E}) = \{s - 3\mathsf{m} = s - \mathsf{M} = 0\} \cup \{s - 3\mathsf{m} = s - 3\mathsf{M} = 0\},$$
$$\nabla_{\texttt{dbox}}(\mathcal{E}) = \{(4\mathsf{mM} - \mathsf{m}s - 4\mathsf{m}t + st)\big(144\mathsf{m}^2\mathsf{M}^2 - 72\mathsf{m}^2\mathsf{M}s - 96\mathsf{m}^2\mathsf{M}t + 9\mathsf{m}^2 s^2 + 24\mathsf{m}^2 st + 16\mathsf{m}^2 t^2 - 96\mathsf{mM}^3 + 24\mathsf{mM}^2 s + 16\mathsf{mM}^2 t + 40\mathsf{mM}st - 10\mathsf{m}s^2 t - 8\mathsf{m}st^2 + 16\mathsf{M}^4 - 8\mathsf{M}^2 st + s^2t^2\big) = 0\}.$$

Thm. 2 in [math-ph/2109.08036 with Telen]

State-of-the-art in Landauology:

The envelope diagram



$$\Delta_{\mathtt{env}}(\mathcal{E}) = \prod_{i=1}^{5} \Delta_{\mathtt{env},i},$$

degree-45 curve

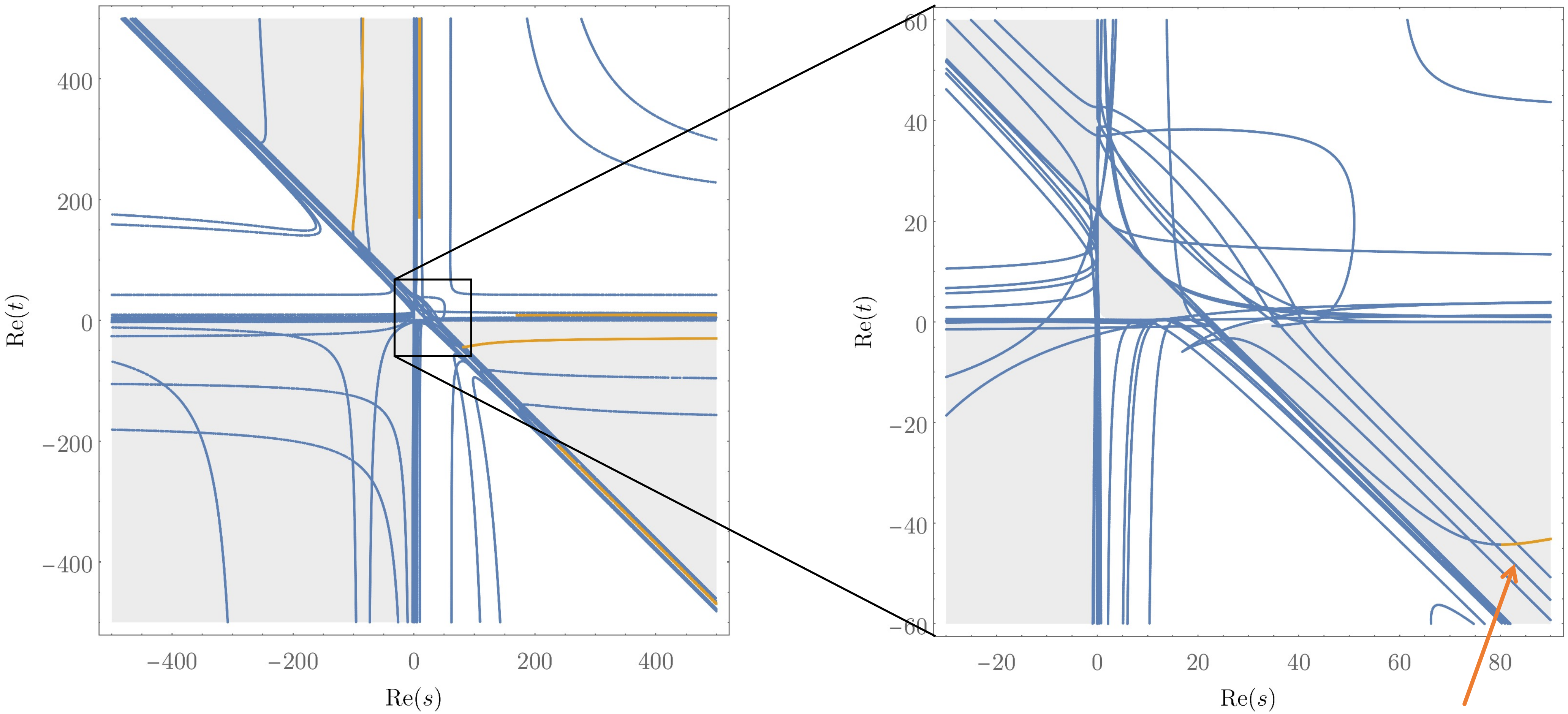
$$
\begin{aligned}
\Delta_{\mathrm{env},1} = {} & -16s\mathrm{M}^7 - 432\mathrm{m}^2\mathrm{M}^6 + 20s^2\mathrm{M}^6 + 192\mathrm{m}s\mathrm{M}^6 + 4st\mathrm{M}^6 + 1728\mathrm{m}^3\mathrm{M}^5 - 8s^3\mathrm{M}^5 - 240\mathrm{m}s^2\mathrm{M}^5 \\
& - 48\mathrm{m}^2s\mathrm{M}^5 + 216\mathrm{m}^2t\mathrm{M}^5 - 8s^2t\mathrm{M}^5 - 72\mathrm{m}st\mathrm{M}^5 - 2592\mathrm{m}^4\mathrm{M}^4 + s^4\mathrm{M}^4 + 96\mathrm{m}s^3\mathrm{M}^4 + 492\mathrm{m}^2s^2\mathrm{M}^4 \\
& - 27\mathrm{m}^2t^2\mathrm{M}^4 + s^2t^2\mathrm{M}^4 + 6\mathrm{m}st^2\mathrm{M}^4 - 1280\mathrm{m}^3s\mathrm{M}^4 - 864\mathrm{m}^3t\mathrm{M}^4 + 2s^3t\mathrm{M}^4 + 134\mathrm{m}s^2t\mathrm{M}^4 - 84\mathrm{m}^2st\mathrm{M}^4 \\
& + 1728\mathrm{m}^5\mathrm{M}^3 - 12\mathrm{m}s^4\mathrm{M}^3 - 240\mathrm{m}^2s^3\mathrm{M}^3 - 128\mathrm{m}^3s^2\mathrm{M}^3 + 108\mathrm{m}^3t^2\mathrm{M}^3 - 28\mathrm{m}s^2t^2\mathrm{M}^3 + 48\mathrm{m}^2st^2\mathrm{M}^3 \\
& + 2448\mathrm{m}^4s\mathrm{M}^3 + 1296\mathrm{m}^4t\mathrm{M}^3 - 40\mathrm{m}s^3t\mathrm{M}^3 - 408\mathrm{m}^2s^2t\mathrm{M}^3 + 1232\mathrm{m}^3st\mathrm{M}^3 - 432\mathrm{m}^6\mathrm{M}^2 + 30\mathrm{m}^2s^4\mathrm{M}^2 \\
& + 224\mathrm{m}^3s^3\mathrm{M}^2 + 2\mathrm{m}s^2t^3\mathrm{M}^2 - 6\mathrm{m}^2st^3\mathrm{M}^2 - 468\mathrm{m}^4s^2\mathrm{M}^2 - 162\mathrm{m}^4t^2\mathrm{M}^2 + 4\mathrm{m}s^3t^2\mathrm{M}^2 + 136\mathrm{m}^2s^2t^2\mathrm{M}^2 \\
& - 468\mathrm{m}^3st^2\mathrm{M}^2 - 1728\mathrm{m}^5s\mathrm{M}^2 - 864\mathrm{m}^5t\mathrm{M}^2 + 2\mathrm{m}s^4t\mathrm{M}^2 + 156\mathrm{m}^2s^3t\mathrm{M}^2 + 156\mathrm{m}^3s^2t\mathrm{M}^2 - 2052\mathrm{m}^4st\mathrm{M}^2 \\
& - 28\mathrm{m}^3s^4\mathrm{M} - 72\mathrm{m}^4s^3\mathrm{M} - 20\mathrm{m}^2s^2t^3\mathrm{M} + 76\mathrm{m}^3st^3\mathrm{M} + 432\mathrm{m}^5s^2\mathrm{M} + 108\mathrm{m}^5t^2\mathrm{M} - 32\mathrm{m}^2s^3t^2\mathrm{M} \\
& - 48\mathrm{m}^3s^2t^2\mathrm{M} + 576\mathrm{m}^4st^2\mathrm{M} + 432\mathrm{m}^6s\mathrm{M} + 216\mathrm{m}^6t\mathrm{M} - 12\mathrm{m}^2s^4t\mathrm{M} - 136\mathrm{m}^3s^3t\mathrm{M} + 288\mathrm{m}^4s^2t\mathrm{M} \\
& + 1080\mathrm{m}^5st\mathrm{M} + 9\mathrm{m}^4s^4 + \mathrm{m}^2s^2t^4 - 4\mathrm{m}^3st^4 + 2\mathrm{m}^2s^3t^3 + 6\mathrm{m}^3s^2t^3 - 54\mathrm{m}^4st^3 - 108\mathrm{m}^6s^2 - 27\mathrm{m}^6t^2 \\
& + \mathrm{m}^2s^4t^2 + 20\mathrm{m}^3s^3t^2 - 45\mathrm{m}^4s^2t^2 - 162\mathrm{m}^5st^2 + 10\mathrm{m}^3s^4t + 18\mathrm{m}^4s^3t - 162\mathrm{m}^5s^2t - 108\mathrm{m}^6st,
\end{aligned}
$$

$$
\Delta_{\mathrm{env},2} = \Delta_{\mathrm{env},1}|_{s\leftrightarrow t}, \qquad \Delta_{\mathrm{env},3} = \Delta_{\mathrm{env},1}|_{t\to u}.
$$

$$
\Delta_{\mathrm{env},4} = 4\mathrm{m}^3\sigma_3^2 + 64\mathrm{m}^2\mathrm{M}^3(\mathrm{m}-\mathrm{M})^4 - \sigma_3\left(27\mathrm{m}^4 - 2\mathrm{m}^2\mathrm{M}^2 + 8\mathrm{m}\mathrm{M}^3 - \mathrm{M}^4\right)(\mathrm{m}-\mathrm{M})^2.
$$

$$
\begin{aligned}
\Delta_{\mathrm{env},5} = {} & 4096\mathrm{M}^{12} - 65536\mathrm{m}\mathrm{M}^{11} + 475136\mathrm{m}^2\mathrm{M}^{10} - 2048\sigma_2\mathrm{M}^{10} - 2064384\mathrm{m}^3\mathrm{M}^9 + 24576\mathrm{m}\sigma_2\mathrm{M}^9 + 2048\sigma_3\mathrm{M}^9 \\
& + 5988352\mathrm{m}^4\mathrm{M}^8 + 256\sigma_2^2\mathrm{M}^8 - 147456\mathrm{m}^2\sigma_2\mathrm{M}^8 + 14336\mathrm{m}\sigma_3\mathrm{M}^8 - 12222464\mathrm{m}^5\mathrm{M}^7 - 2048\mathrm{m}\sigma_2^2\mathrm{M}^7 \\
& + 606208\mathrm{m}^3\sigma_2\mathrm{M}^7 - 313344\mathrm{m}^2\sigma_3\mathrm{M}^7 - 512\sigma_2\sigma_3\mathrm{M}^7 + 18006016\mathrm{m}^6\mathrm{M}^6 + 15360\mathrm{m}^2\sigma_2^2\mathrm{M}^6 + 128\sigma_3^2\mathrm{M}^6 \\
& - 1847296\mathrm{m}^4\sigma_2\mathrm{M}^6 + 1783808\mathrm{m}^3\sigma_3\mathrm{M}^6 - 8704\mathrm{m}\sigma_2\sigma_3\mathrm{M}^6 - 19300352\mathrm{m}^7\mathrm{M}^5 - 79872\mathrm{m}^3\sigma_2^2\mathrm{M}^5 \\
& + 2560\mathrm{m}\sigma_3^2\mathrm{M}^5 + 4096000\mathrm{m}^5\sigma_2\mathrm{M}^5 - 5031936\mathrm{m}^4\sigma_3\mathrm{M}^5 + 82432\mathrm{m}^2\sigma_2\sigma_3\mathrm{M}^5 + 14946304\mathrm{m}^8\mathrm{M}^4 \\
& - 1024\mathrm{m}^2\sigma_2^3\mathrm{M}^4 + 230912\mathrm{m}^4\sigma_2^2\mathrm{M}^4 + 27136\mathrm{m}^2\sigma_3^2\mathrm{M}^4 + 32\sigma_2\sigma_3^2\mathrm{M}^4 - 6348800\mathrm{m}^6\sigma_2\mathrm{M}^4 \\
& + 7350272\mathrm{m}^5\sigma_3\mathrm{M}^4 + 1280\mathrm{m}\sigma_2^2\sigma_3\mathrm{M}^4 - 390656\mathrm{m}^3\sigma_2\sigma_3\mathrm{M}^4 - 8159232\mathrm{m}^9\mathrm{M}^3 + 4096\mathrm{m}^3\sigma_2^3\mathrm{M}^3 \\
& - 32\sigma_3^3\mathrm{M}^3 - 374784\mathrm{m}^5\sigma_2^2\mathrm{M}^3 - 438784\mathrm{m}^3\sigma_3^2\mathrm{M}^3 - 1408\mathrm{m}\sigma_2\sigma_3^2\mathrm{M}^3 + 6602752\mathrm{m}^7\sigma_2\mathrm{M}^3 \\
& - 4241408\mathrm{m}^6\sigma_3\mathrm{M}^3 - 4096\mathrm{m}^2\sigma_2^2\sigma_3\mathrm{M}^3 + 1171968\mathrm{m}^4\sigma_2\sigma_3\mathrm{M}^3 + 2981888\mathrm{m}^{10}\mathrm{M}^2 - 6144\mathrm{m}^4\sigma_2^3\mathrm{M}^2 \\
& + 1184\mathrm{m}\sigma_3^3\mathrm{M}^2 + 343040\mathrm{m}^6\sigma_2^2\mathrm{M}^2 + 1443456\mathrm{m}^4\sigma_3^2\mathrm{M}^2 - 5440\mathrm{m}^2\sigma_2\sigma_3^2\mathrm{M}^2 - 4360192\mathrm{m}^8\sigma_2\mathrm{M}^2 \\
& - 1185792\mathrm{m}^7\sigma_3\mathrm{M}^2 + 43520\mathrm{m}^3\sigma_2^2\sigma_3\mathrm{M}^2 - 2141696\mathrm{m}^5\sigma_2\sigma_3\mathrm{M}^2 - 655360\mathrm{m}^{11}\mathrm{M} + 4096\mathrm{m}^5\sigma_2^3\mathrm{M} \\
& - 7072\mathrm{m}^2\sigma_3^3\mathrm{M} - 165888\mathrm{m}^7\sigma_2^2\mathrm{M} - 242688\mathrm{m}^5\sigma_3^2\mathrm{M} + 67200\mathrm{m}^3\sigma_2\sigma_3^2\mathrm{M} + 1646592\mathrm{m}^9\sigma_2\mathrm{M} \\
& + 1425408\mathrm{m}^8\sigma_3\mathrm{M} - 129024\mathrm{m}^4\sigma_2^2\sigma_3\mathrm{M} + 2366976\mathrm{m}^6\sigma_2\sigma_3\mathrm{M} + 65536\mathrm{m}^{12} + \sigma_3^4 - 1024\mathrm{m}^6\sigma_2^3 \\
& + 13024\mathrm{m}^3\sigma_3^3 - 48\mathrm{m}\sigma_2\sigma_3^3 + 33024\mathrm{m}^8\sigma_2^2 - 3433728\mathrm{m}^6\sigma_3^2 + 768\mathrm{m}^2\sigma_2^2\sigma_3^2 - 149472\mathrm{m}^4\sigma_2\sigma_3^2 \\
& - 270336\mathrm{m}^{10}\sigma_2 + 458752\mathrm{m}^9\sigma_3 - 4096\mathrm{m}^3\sigma_2^3\sigma_3 + 137472\mathrm{m}^5\sigma_2^2\sigma_3 - 1276416\mathrm{m}^7\sigma_2\sigma_3.
\end{aligned}
$$

$\sigma_2 := st + tu + us, \qquad \sigma_3 := stu.$ ← symmetric kinematic invariants



necessarily on the physical sheet ($\alpha_e \in \mathbb{R}_+$)

Multiple other applications:

- Landau polytopes, $\mathcal{A}$-discriminants, toric resultants, …
- Coleman–Norton analysis of physical singularities
- Counting the number of master integrals

[math-ph/2109.08036 with Telen]

Summary:

- Landau discriminants determine singularities of Feynman integrals
- Computations using numerical algebraic geometry methods
- Implemented in an open-source package `Landau.jl`

Thank you!