Feynman Integrals and Intersection Theory

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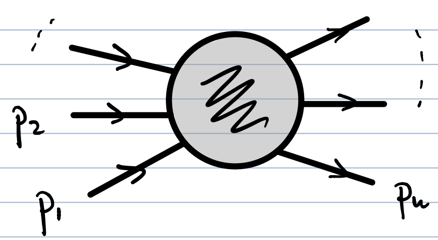
hep-th/2002.10476 review

based on work with Mastrolia (Padova); Frellesvig (Copenhagen), Gasparotto, Laporta, Mandal, Mattiazzi; Pokraka (McGill)



= extremely complicated functions of kinematic invariants

$$s_{ij} = (p_i + p_j)^2,$$

even in perturbation theory

General philosophy to make progress:

promote s_{ij}'s to complex variables

Complex analysis => Physics

bounds

crossing symmetry

dispersion relations

anomalous thresholds

Regge behaviour

unitarity methods

PREFACE

One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane. From the early

[Eden et al. "The Analytic S-Matrix"]

But of course the kinematic space is not *just* the complex plane; it has an intricate geometry!

Algebraic geometry / topology $\Rightarrow$ Physics

vector space of Feynman integrals

basis expansion

differential equations

This idea was briefly considered in the early years of the S-matrix theory

[Fotiadi, Froissart, Lascoux, Pham '60s]

new mathematics [Aomoto et al. '70–90s]

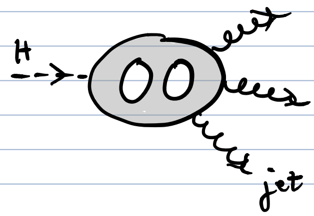
dimensional regularization [Bollini–Giambiagi, 't Hooft–Veltman '72]

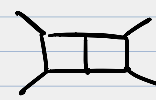
Modern revival using "intersection theory": this talk

[with Mastrolia et al. '18–]

The types of problems we'd like to study generally fall into two categories:

→ Mostly practical: state-of-the-art SM/QCD computations

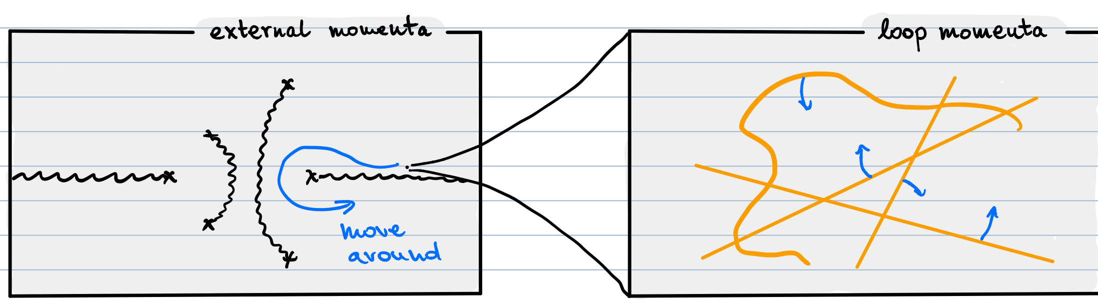


$= \sum_{i=1}^{10^5}$ coefficient$_i$  $_i$

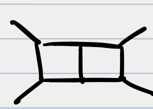
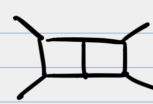
naively $\mathcal{O}(10^5)$ scalar Feynman diagrams to evaluate!

But it turns out these are hugely redundant, and only $\mathcal{O}(10^3)$ are actually independent ⇝ basis?

→ Mostly formal: analyticity structure, number-theoretic aspects, geometry, ...



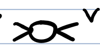
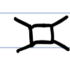
All encapsulated by differential equations, schematically:

∂_s  $_i$ $= \Omega_{ij}$  $_j$

Outline

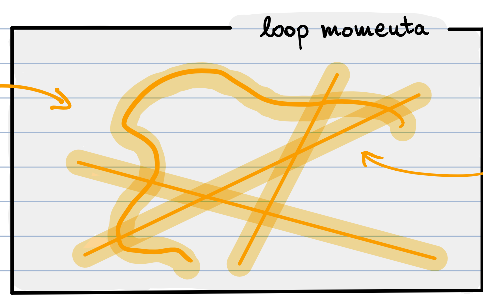
→ Review how to manifest singularities of Feynman integrals: geometry of the loop momentum space.

→ Model the vector space of Feynman integrals as a cohomology group of the loop momentum space.

→ Introduce a "scalar product" between Feynman integrals, $\langle$  $^{\vee} \,|\,$  $\rangle$ = rational function.

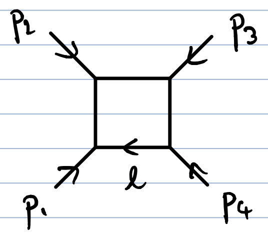
What this might suggest is a shift in our understanding

finite parts of the integral should be governed by behavior away from singularities

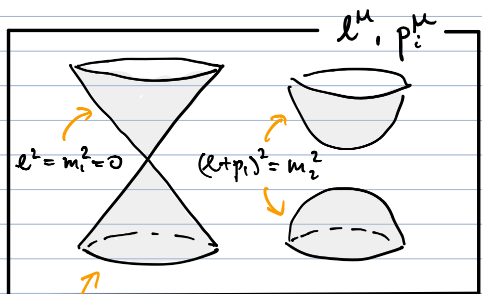


scalar products localize on the singularities

Let's start with a simple example of a Feynman diagram:

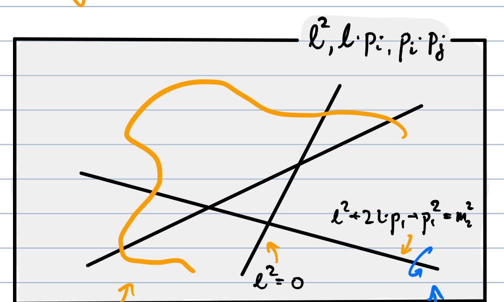


$$= \frac{d^4\ell}{(\ell^2-m_1^2)((\ell+p_1)^2-m_2^2)(\;\cdots\;)(\;\cdots\;)}$$



manifest Lorentz invariance and UV/IR singularities

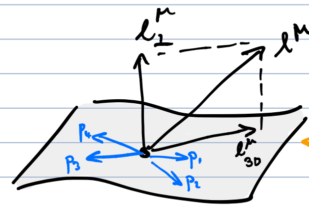
remove singularities



UV/IR singularities

unitarity cuts

In dimensional regularization: $\ell^\mu = \ell^\mu_{3D} + \ell^\mu_\perp$



4 Lorentz vectors can be embedded in $\mathbb{R}^{1,2}$

Loop integrand takes the form:

$$d^D\ell \; f(\ell^2, \ell\cdot p_i, p_i\cdot p_j) = d^3\ell_{3D}\, d^{D-3}\ell_\perp \; f(\overbrace{\ell^2_{3D}+\ell^2_\perp}^{\ell^2}, \ell_{3D}\cdot p_i, p_i\cdot p_j)$$

only the norm of $\ell^\mu_\perp$ matters

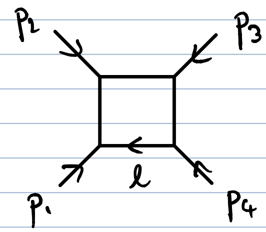
$$= \#\, d^3\ell_{3D} \frac{d(\ell^2_\perp)}{(\ell^2_\perp)^{(4-D)/4}} f(\ell^2, \ell\cdot p_i, p_i\cdot p_j)$$

Jacobian

Finally, we notice that $\ell^2_\perp = \ell^2 - \ell^2_{3D}$

$$= \# \begin{vmatrix} \ell^2 & \ell\cdot p_1 & \ell\cdot p_2 & \ell\cdot p_3 \\ \ell\cdot p_1 & p_1^2 & p_1\cdot p_2 & p_1\cdot p_3 \\ \ell\cdot p_2 & p_1\cdot p_2 & p_2^2 & p_2\cdot p_3 \\ \ell\cdot p_3 & p_1\cdot p_3 & p_2\cdot p_3 & p_3^2 \end{vmatrix}$$

and hence everything's expressed in terms of ℓ^2, $\ell \cdot p_i$:



$$= \frac{\# \; d(\ell^2) \wedge d(\ell \cdot p_1) \wedge d(\ell \cdot p_2) \wedge d(\ell \cdot p_3)}{(\ell_\perp^2)^{\frac{1}{2}(1+\epsilon)} (\ell^2 - m_1^2)((\ell + p_1)^2 - m_2^2)(\;\cdots\;)(\;\cdots\;)}$$

continue to $D = 4 - 2\varepsilon$

Completely general (take $n \leq 5$):

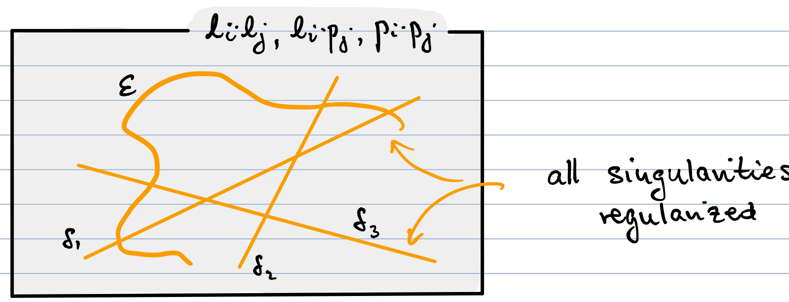
$L(n-1) + \frac{L(L+1)}{2}$ variables

$$\frac{\prod_{i=1}^{L} d^{4-2\epsilon} \ell_i}{\prod_j (q_j^2 - m_j^2)^{1+\delta_j}} = \# \frac{\prod_{i,j} d(\ell_i \cdot \ell_j) \prod_{i,j} d(\ell_i \cdot p_j)}{\begin{vmatrix} \ell_i \cdot \ell_j & \ell_i \cdot p_j \\ \ell_i \cdot p_j & p_i \cdot p_j \end{vmatrix}^{\# \varepsilon} \prod_j (q_j^2 - m_j^2)^{1+\delta_j}}$$

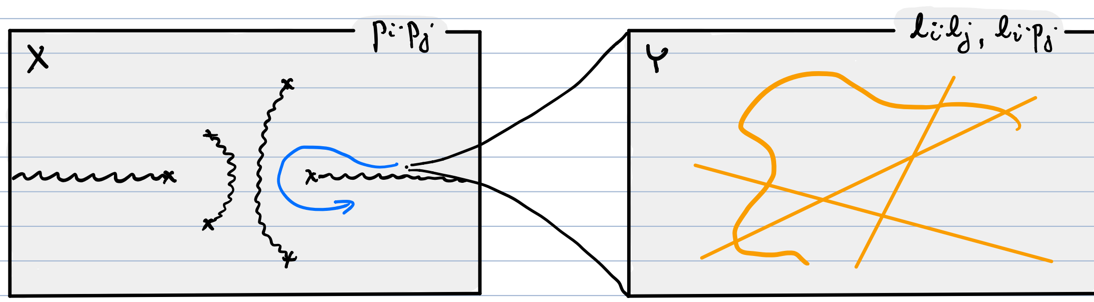
[Baikov '96]

additional "analytic regularization"

$\delta_j \to 0$ [Speer '69]



We should really distinguish between external and internal invariants (fiber bundle) and complexify:



$\dim_{\mathbb{C}} X = 3n-10$ $\qquad$ $\dim_{\mathbb{C}} Y = L(n-1) + \frac{L(L+1)}{2}$

Integrating out the internal space Y leaves us with the Feynman integral $I(p_i \cdot p_j)$ on X. It satisfies a differential equation

$$\left(\sum_{i,j} \frac{\partial^x}{\partial(p_i \cdot p_j)\,\partial(\dots)\dots} + \sum_{i,j} \frac{\partial^{x-1}}{\partial(p_i \cdot p_j)\,\partial(\dots)} + \dots \right) I = 0.$$

high-order

It turns out to be more convenient to linearize the differential equation into a coupled system of χ equations: [Henn et al. '13–]

$$D \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_\chi \end{bmatrix} = \Omega \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_\chi \end{bmatrix}$$

Kinematic differential

$$D = \sum_{i,j} d(p_i \cdot p_j) \frac{\partial}{\partial (p_i \cdot p_j)}$$

χ-by-χ matrix 1-form

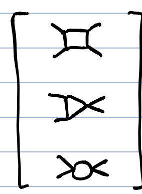
$$\Omega = \sum_{i,j} d(p_i \cdot p_j)\, \Omega^{(i,j)}$$

(tells us where the singularities are)

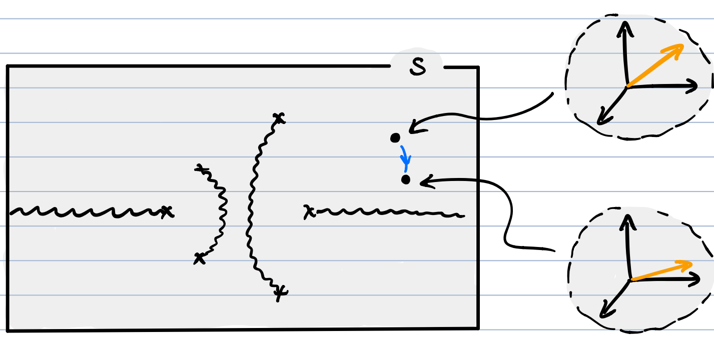
In fact, χ will turn out to be a topological invariant of Y.

For example, massless box has $\chi = 3$:

$$\vec{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} =$$



$(s, t, \varepsilon) \in$ some vector space

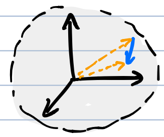
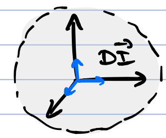


$= \vec{I}$

$= \vec{I} + D\vec{I}$

$$D = ds \frac{\partial}{\partial s} + dt \frac{\partial}{\partial t}$$

dual basis

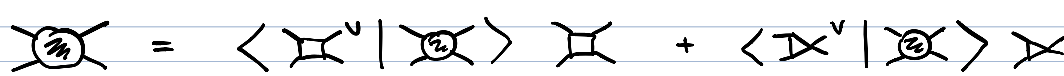
$D\vec{I} =$  $=$  $= \underbrace{\langle \vec{I}^\vee | D\vec{I} \rangle}_{\Omega} \vec{I}$

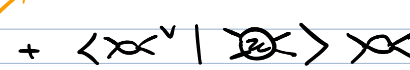
project $D\vec{I}$ onto the original basis $\vec{I}$

Hence we need to find out :

- What exactly is the vector space in which $\vec{I}$ live?
- What's the dual space of $\vec{I}^\vee$?
- How to define & compute the scalar product $\langle \vec{I}^\vee | \vec{I} \rangle$ between Feynman integrals ?

Completely general manipulations, e.g.,

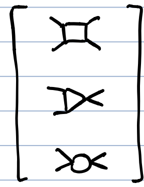




coefficients of the basis expansion

Going back to our example:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} =$$



$$= \int \underbrace{\frac{1}{(\ell_\perp^2)^{\frac{1}{2}(1+\varepsilon)} \, \mathcal{P}_1^{\delta_1} \mathcal{P}_2^{\delta_2} \mathcal{P}_3^{\delta_3} \mathcal{P}_4^{\delta_4}}}_{\text{common factor } e^W} \underbrace{\begin{bmatrix} \dfrac{d^4\mu}{\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3 \mathcal{P}_4} \\ \dfrac{d^4\mu}{\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3} \\ \dfrac{d^4\mu}{\mathcal{P}_1 \mathcal{P}_3} \end{bmatrix}}_{\text{specific diagram } \varphi_i}$$

where $d^4\mu = d(\ell_\perp^2) \wedge d(\ell \cdot p_1) \wedge d(\ell \cdot p_2) \wedge d(\ell \cdot p_3)$

$\mathcal{P}_1 = \ell^2 - m_1^2 , \quad \mathcal{P}_2 = (\ell + p_1)^2 - m_2^2 , \quad \ldots$

Hence the integrals take the form:

$$\boxed{I_i = \int_\Gamma e^W \varphi_i}$$

top holomorphic form on Y

no boundary terms since all singularities are regulated

(all minors of the Gram determinant are positive)

What's the space of all possible φ_i we can write down?

Total derivatives don't change the answer:

$$I_i = \int e^W \overset{m\text{-form}}{\varphi_i} + d(e^W \overset{(m-1)\text{-form}}{\xi}) \qquad m = \dim_{\mathbb{C}} Y.$$

$$= \int e^W (\varphi_i + d\xi + dW \wedge \xi)$$

$$= \int e^W (\varphi_i + \nabla_{dW} \xi), \qquad \text{where } \nabla_{dW} = d + dW\wedge$$

$$(\nabla_{dW}^2 = 0 \text{ integrable})$$

Therefore we have equivalence classes:

$$\varphi_i \sim \varphi_i + \nabla_{dW} \xi$$

This is a space known to mathematicians, called twisted cohomology, $H^m(Y, \nabla_{dW})$.

How many distinct equivalence (cohomology) classes of Feynman integrals can we write down?

de Rham cohomology

$$\dim H^0(Y) = \#$$
$$\dim H^1(Y) = \#$$
$$\vdots$$
$$\dim H^m(Y) = \#$$
$$\vdots$$

twisted cohomology

$$\dim H^0(Y, \nabla_{dW}) = 0$$
$$\dim H^1(Y, \nabla_{dW}) = 0$$
$$\vdots$$
$$\dim H^m(Y, \nabla_{dW}) = \#$$
$$\vdots$$

the only non-vanishing!

[Aomoto '75]

Their alternating sum is a topological invariant:

$$\chi(Y) = \sum_k (-1)^k \dim H^k(Y, \nabla_{dW}) = (-1)^m \dim H^m(Y, \nabla_{dW}),$$

Euler characteristic

So the number of basis Feynman diagrams $= (-1)^m \chi(Y)$

$= \#$ of critical pts. of $\mathrm{Re}(W)$.

We will define the dual space of Feynman integrals to be

$$\varphi_i^\vee \sim \varphi_i^\vee + \nabla_{-dW}\,\xi \;:\qquad H^n(Y, \nabla_{-dW})$$

minus sign

$$\nabla_{-dW} = d - dW\wedge$$

This is roughly considering Feynman integrals in $4+2\varepsilon$ instead of $4-2\varepsilon$ dimensions and $\delta_j \to -\delta_j$.

Now we need the scalar product, which is tentatively given by the "intersection number":

$$\langle \varphi^\vee | \varphi \rangle = \int_Y \varphi^\vee \wedge \varphi \, .$$

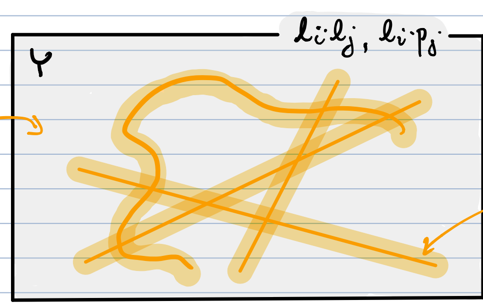
non-compact space

two holomorphic top forms

in the bulk $\varphi^\vee \wedge \varphi = 0$

so

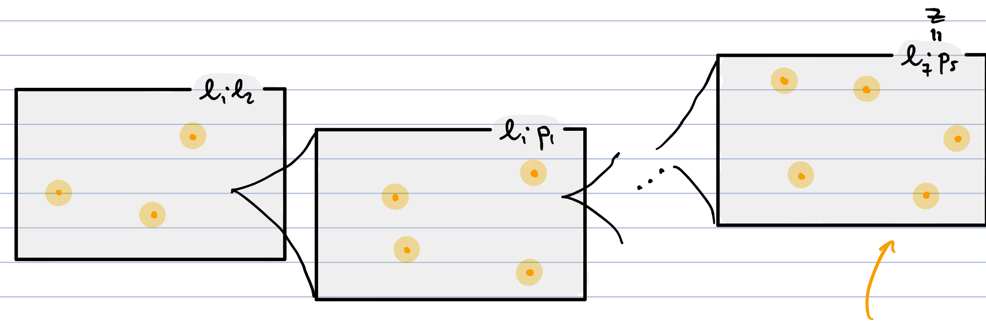
$$\int_{\text{bulk of } Y} \varphi^\vee \wedge \varphi = 0$$



near singularities there's a "0/0" problem

$$\int_{\text{boundary of } Y} \varphi^\vee \wedge \varphi \neq 0$$

The strategy for regularization is to first fiber φ into m one-dimensional spaces: [SM '19]

$\varphi =$ 

all singularities are points

Let's just compute the intersection number on the final fiber:

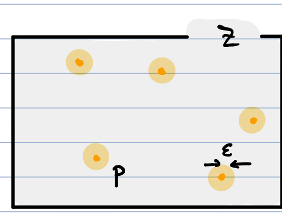
impose compact support

$$\langle \varphi^\vee | \varphi \rangle = \frac{-1}{2\pi i} \int_{\mathbb{CP}^1 - \bigcup_p \{z=p\}} \varphi^\vee \wedge \varphi_{reg}$$

dz-forms

for convenience

We need to find φ_{reg} in the same equivalence class as φ but having compact support (vanish near all singular points):



$$\varphi_{reg} = \varphi - \nabla_{dw}\left(\sum_p \Theta(|z-p|^2 - \varepsilon^2)\, \nabla_{dw}^{-1} \varphi \right)$$

step fn's vanish outside $\bullet_p$

$$= \varphi \left(1 - \underbrace{\sum_p \Theta(|z-p|^2 - \varepsilon^2)}_{} \right) - \sum_p \delta(|z-p|^2 - \varepsilon^2)\, \nabla_{dw}^{-1} \varphi$$

vanishes near each p

but still $\varphi^\vee \wedge \varphi(\dots) = 0$

Plugging back into the definition:

$$\langle \varphi^\vee | \varphi \rangle = \frac{1}{2\pi i} \int_{\mathbb{CP}^1 - \cup_p \{z=p\}} \varphi^\vee \wedge \sum_p \delta(|z-p|^2 - \varepsilon^2)\, \nabla_{dw}^{-1} \varphi$$

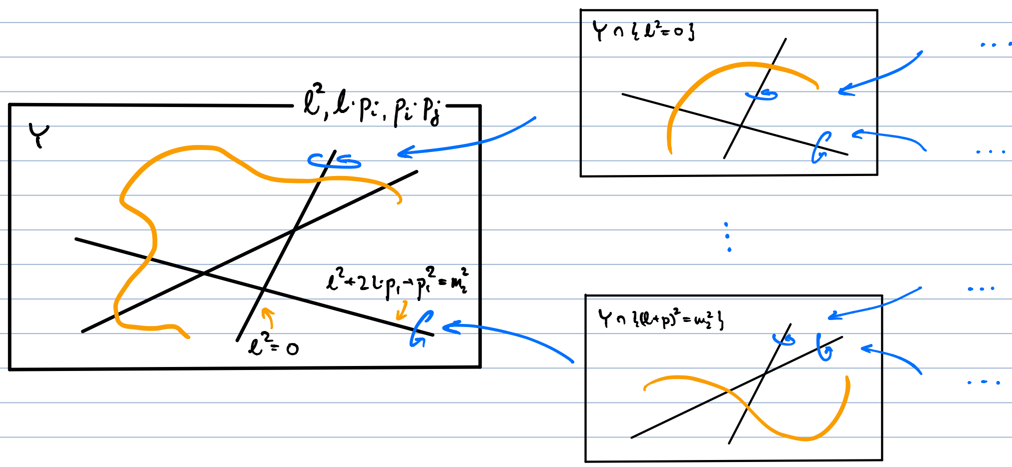
$$= \sum_p \operatorname{Res}_{z=p} \left(\varphi^\vee \, \nabla_{dw}^{-1} \varphi \right),$$

where $\nabla_{dw}^{-1} \varphi = \psi_p$ just means a <u>local</u> solution of the differential equation $\varphi = \nabla_{dw} \psi_p$ near $z = p$.

Therefore it's much easier to compute intersection numbers (scalar products of Feynman integrals) than the integrals themselves!

↑ Scroll up for a recap ↑

Future directions are mostly tied to relaxing the analytic regularization, which would explore the full unitarity structure



Work in progress [SM; Caron-Huot, Pokraka]

Thanks!

Extra notes:

→ How do we actually find the dual basis of integrals?

$$\langle \varphi_i^\vee | \varphi_j \rangle = \delta_{ij}$$

Simply take any dual basis $\Phi_i^\vee$, i.e.,

$$\langle \Phi_i^\vee | \varphi_j \rangle = C_{ij}$$

invertible χ-by-χ matrix

Then $\varphi_i^\vee = \sum_{k=1}^{\chi} C^{-1}_{ik} \, \Phi_k^\vee$, because

$$\langle \varphi_i^\vee | \varphi_j \rangle = \sum_{k=1}^{\chi} C^{-1}_{ik} \langle \Phi_k^\vee | \varphi_j \rangle = \sum_{k=1}^{\chi} C^{-1}_{ik} C_{kj} = \delta_{ij}$$