Evaluating one-loop string amplitudes

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Based on [hep-th/2208.12233]
and [hep-th/2302.12733]
with Lorenz Eberhardt

Surprisingly little is known about scattering amplitudes in string theory

Veneziano amplitude

Polarization dependence $t_8 = s\, p_1{\cdot}\epsilon_2\, p_2{\cdot}\epsilon_1\, \epsilon_3{\cdot}\epsilon_4 + \ldots$

$$\mathcal{A}^{\text{planar}}_{\text{tree}}(s,t) = -t_8 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s - \alpha' t)}$$

Center of mass energy

Momentum transfer

Inverse string tension

Higher-loop contributions

Textbook definition of string amplitudes

Known for low g and n

$$\mathcal{A}_{g,n}(p_1, p_2, \ldots, p_n) \stackrel{?}{=} \int_{\mathcal{M}_{g,n}} (\text{correlation function})$$

or $\Gamma \subset \mathcal{M}_{g,n}$

Moduli space of genus-g Riemann surfaces with n punctures

isn't entirely correct, e.g., not consistent with unitarity (the integration domain isn't known)

The underlying problem is that we formulate string amplitudes on a *Euclidean* worldsheet, but the target space is *Lorentzian*

$\mathbb{R}^{1,9}$
in this talk

(the reason to formulate the theory on a Euclidean worldsheet in the first place is to be able to use CFT technology, manifest UV finiteness, …)

Why hasn't it been a problem before?

Most computations done:

- At tree level

(meromorphic functions)

- At loop level in the $\alpha' \to 0$ or high-energy expansion

(branch cut ambiguities fixed by matching with EFT)

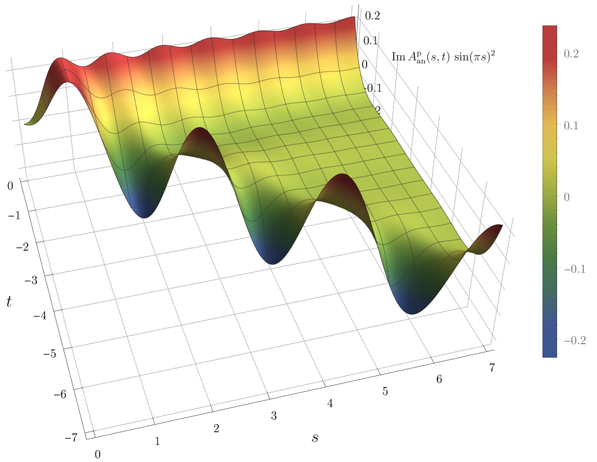
[enormous literature: Green, Schwarz, Gross, Veneziano, Amati, Ciafaloni, Di Vecchia, Koba, Nielsen, D'Hoker, Phong, Martinec, Bern, Dixon, Polyakov, Kosower, Vanhove, Schlotterer, Mafra, Stieberger, Brown, Broedel, Hohenegger, Kleinschmidt, Gerken, Roiban, Lipstein, Mason, Monteiro, ...]

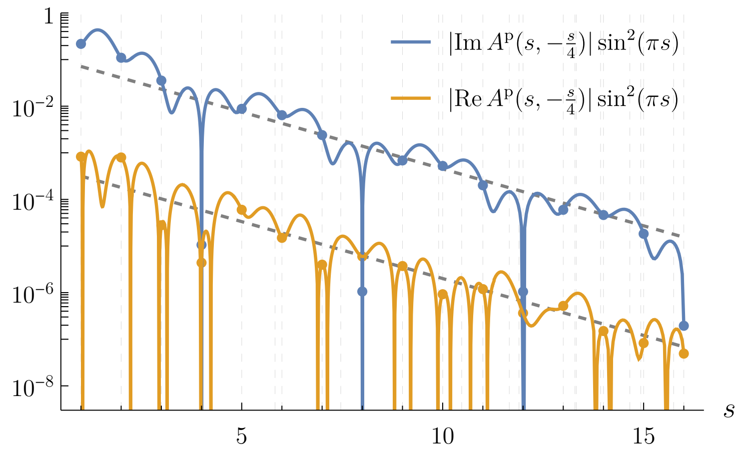
Challenge:

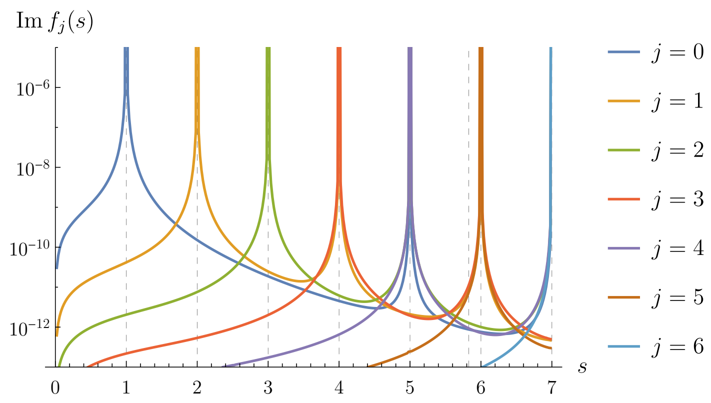
Specifying the external momenta p_i ,

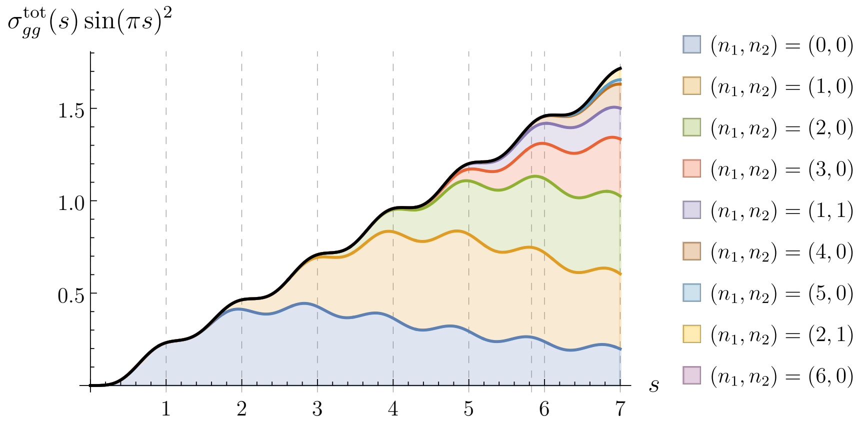
can we compute *any* of the string loop amplitudes?

In this talk we'll do it for one-loop superstring amplitudes



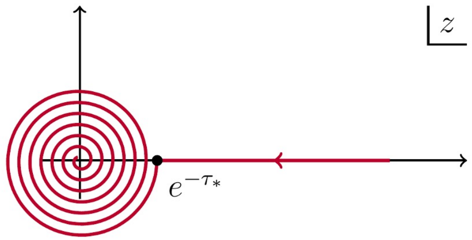




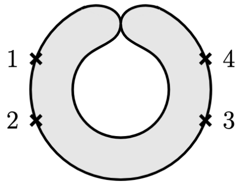


Outline of the talk

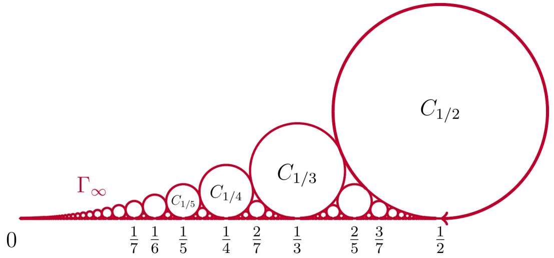
1) From Euclidean to Lorentzian worldsheets



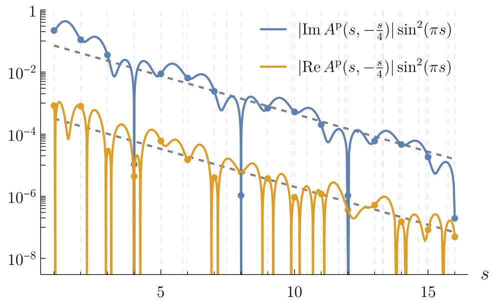
2) Unitarity cuts of the worldsheet



3) Rademacher expansion

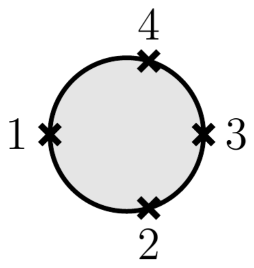


4) Physics of one-loop amplitudes



Let’s start at tree level

($\alpha' = 1$ from now on)



$$= \frac{t_8}{t} \int_0^1 z^{-s-1} (1-z)^{-t} \, \mathrm{d}z$$

$s > 0$ $t < 0$

Length of the neck (Schwinger parameter)

s-channel poles come from $z = \dfrac{z_{12} z_{34}}{z_{13} z_{24}} \approx 0$, so set $z = e^{-\tau}$ and take $\tau \to \infty$

$$t_8 \int_0^\infty e^{\tau s} \left(\# + \# e^{-\tau} + \# e^{-2\tau} + \ldots \right) \mathrm{d}\tau$$

$-\dfrac{1}{s}$	$-\dfrac{\#}{s-1}$	$-\dfrac{\#}{s-2}$
massless	level-1	level-2

Important distinction

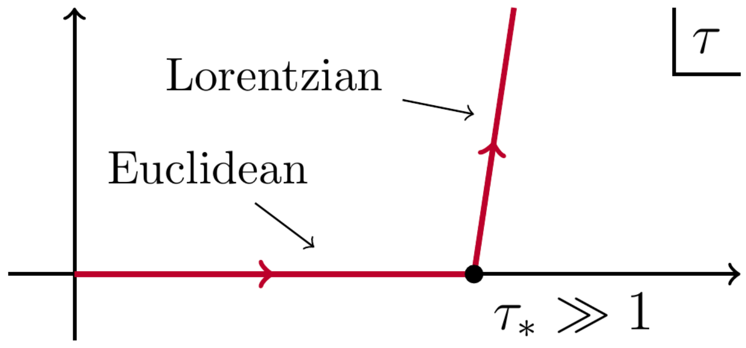
$$\frac{-1}{s-m^2} = \int_0^\infty \mathrm{d}\tau_{\mathrm{E}}\, e^{\tau_{\mathrm{E}}(s-m^2)}$$

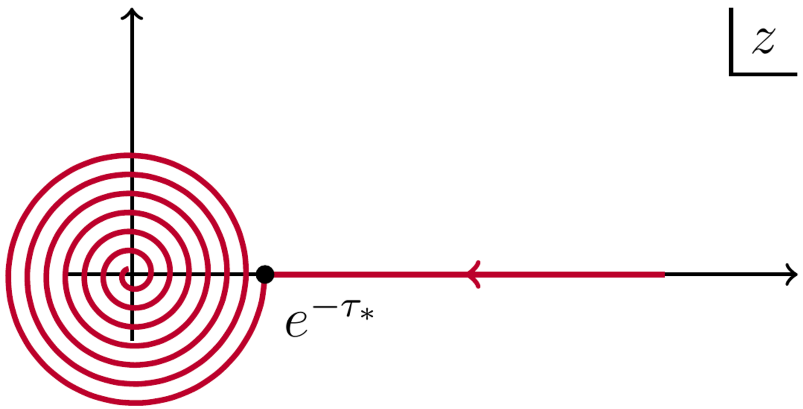
Euclidean proper time

$$\frac{i}{s-m^2} = \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\tau_{\mathrm{L}}\, e^{i\tau_{\mathrm{L}}(s-m^2+i\varepsilon)}$$

Lorentzian proper time

This tells us about the correct integration contour





We can resum $\circlearrowright + e^{-2\pi i s} \circlearrowright + e^{-4\pi i s} \circlearrowright + \ldots = \dfrac{1}{1 - e^{-2\pi i s}} \circlearrowright$

infinite number of string resonances

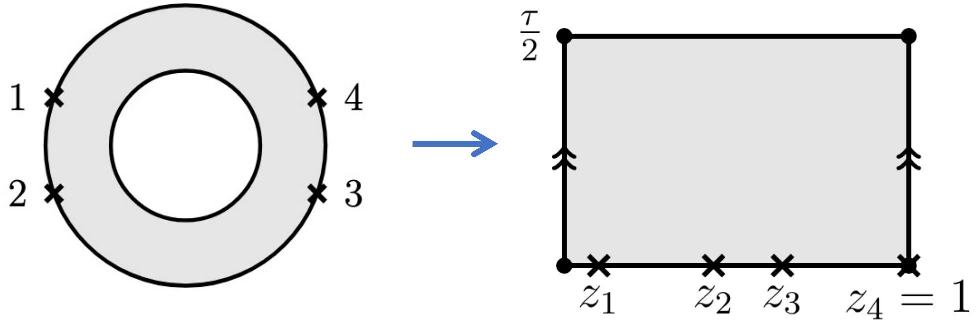
Strategy for finding the contour at higher genus

- Identify local variables $q \sim e^{-(\text{Schwinger parameter})}$
- Continue to Lorentzian signature locally in the moduli space
- Glue everything together

[Witten '13]

Genus-one superstring amplitudes

In this talk we focus on the planar annulus contribution



Modular parameter

$z_{ij} = z_i - z_j$

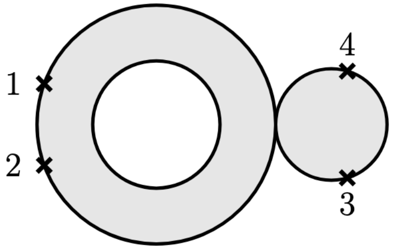
$$\mathcal{A}^{\text{planar}}_{\text{annulus}} \stackrel{?}{=} -i\, t_8 \int_0^{i\infty} \mathrm{d}\tau \int_{0<z_1<z_2<z_3<1} \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)}\right)^{-s} \left(\frac{\vartheta_1(z_{32},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)}\right)^{-t}$$

[Green, Schwarz '82]

Jacobi theta function

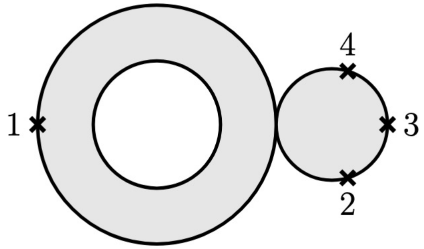
$$\vartheta_1(z,\tau) = i\sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{2\pi i(n-\frac{1}{2})z + \pi i (n-\frac{1}{2})^2\tau}$$

Various degenerations need the Witten $i\varepsilon$



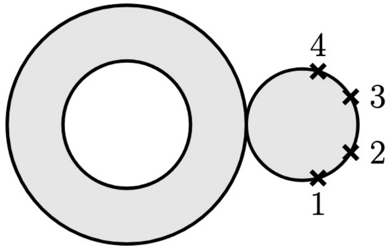
Massive pole exchange

$$q = z_{43}$$



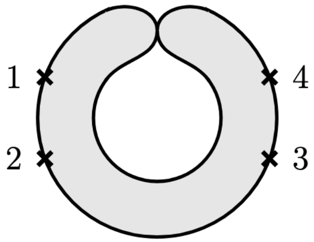
Wave-function renormalization

$$q = z_{42}$$



Tadpole

$$q = z_{41}$$

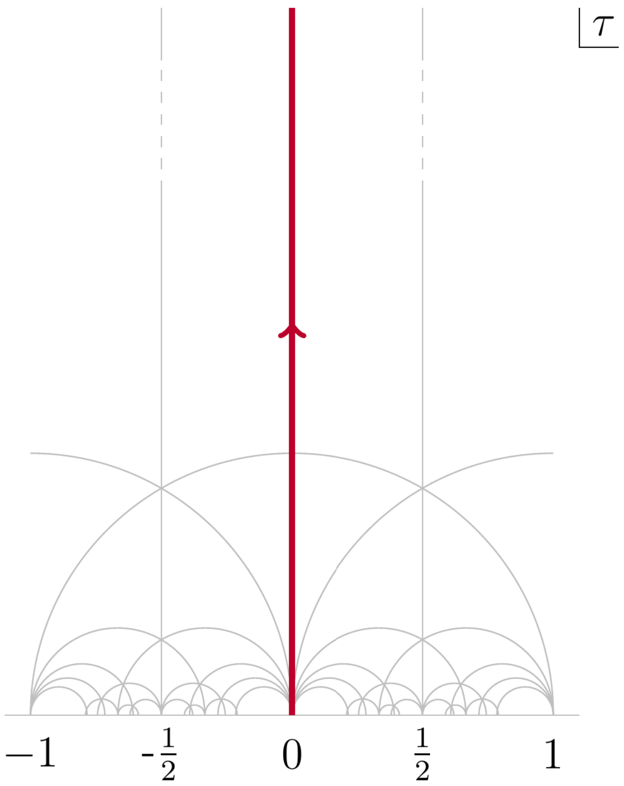


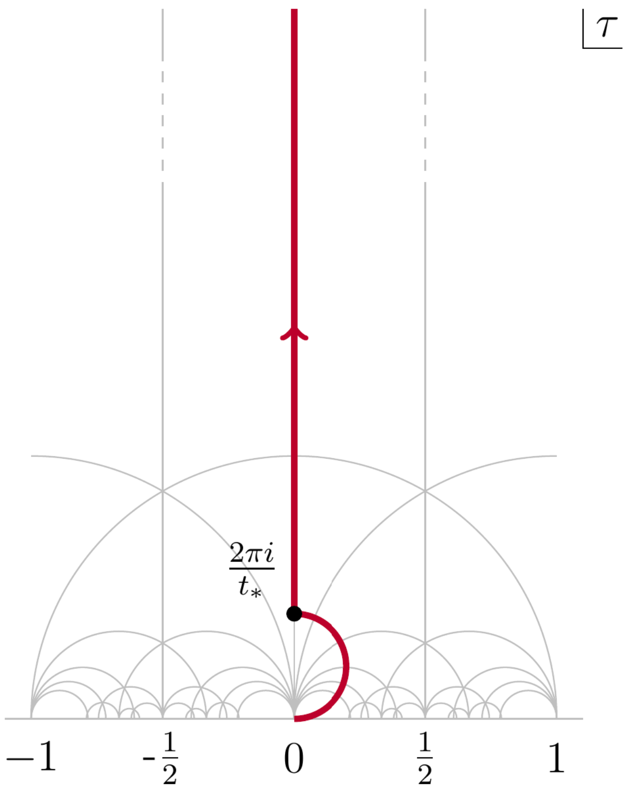
Non-separating degeneration

$$q = e^{-\frac{2\pi i}{\tau}}$$

Unitarity cuts

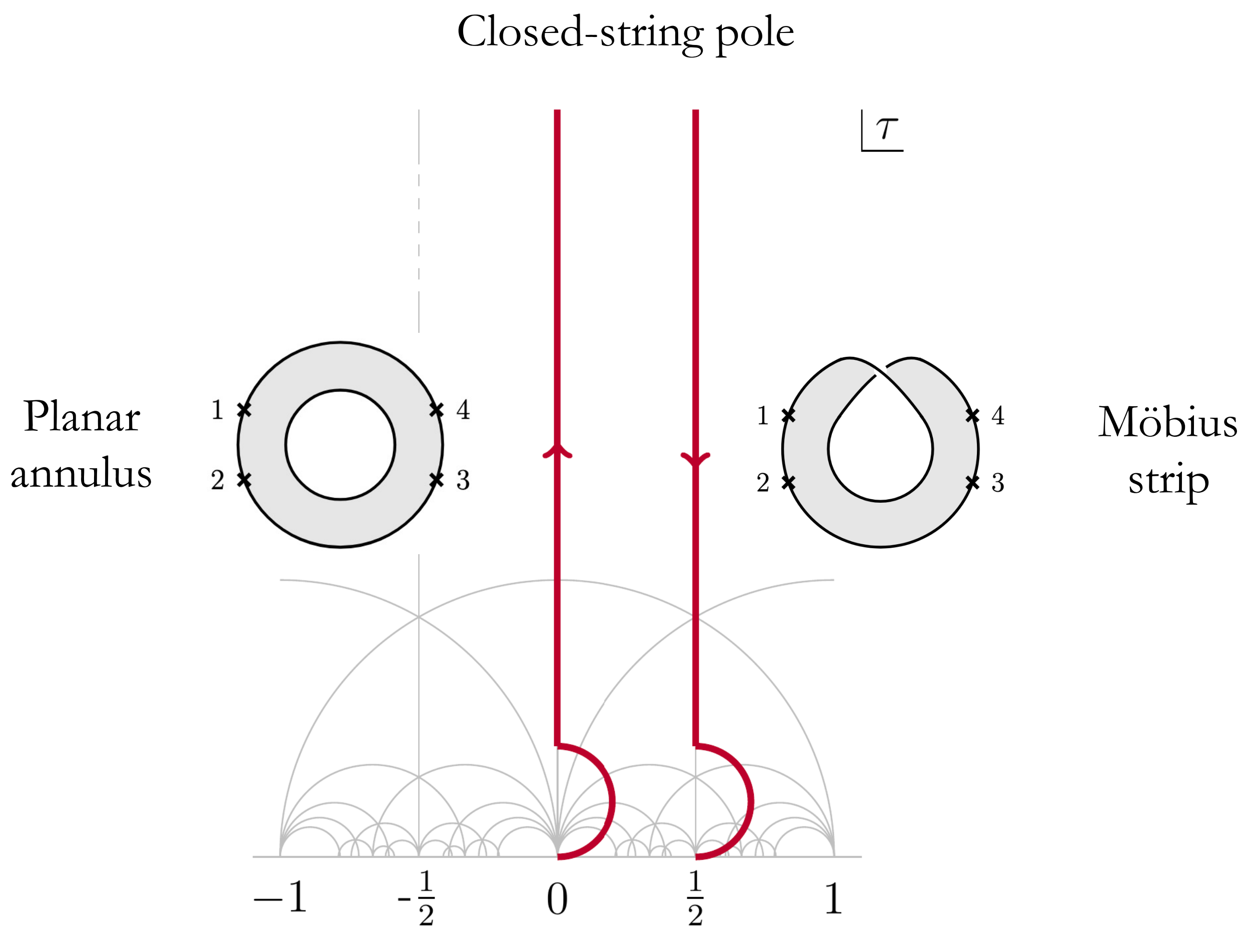
Let's focus on the contour in the fundamental domain, $\tau = \dfrac{2\pi i}{t_* + it}$





Approach the essential singularity from the right

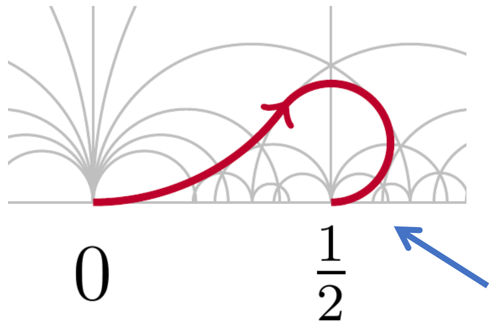
Adding the other planar contribution: Möbius strip



Our proposal for the correct integration contour

(similar for non-planar amplitudes)

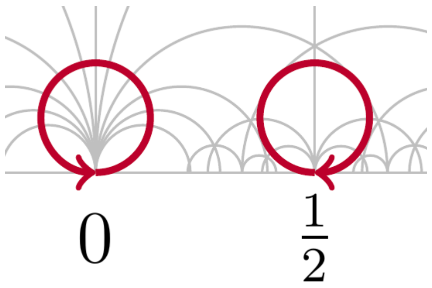
Precise shape doesn't matter



Approach 0 and ½ from the right

For the imaginary part we only need

Size of the circles doesn't matter



They'll give as unitarity cuts of the planar annulus and the Möbius strip

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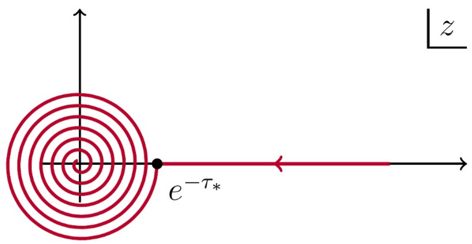

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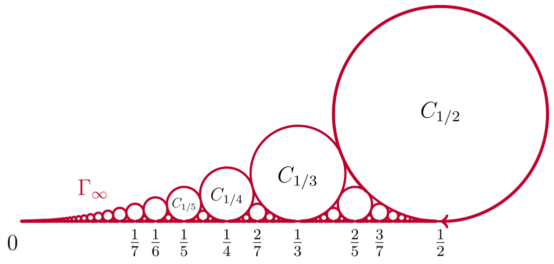

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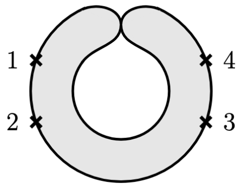

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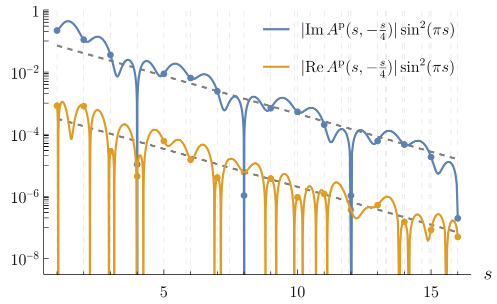
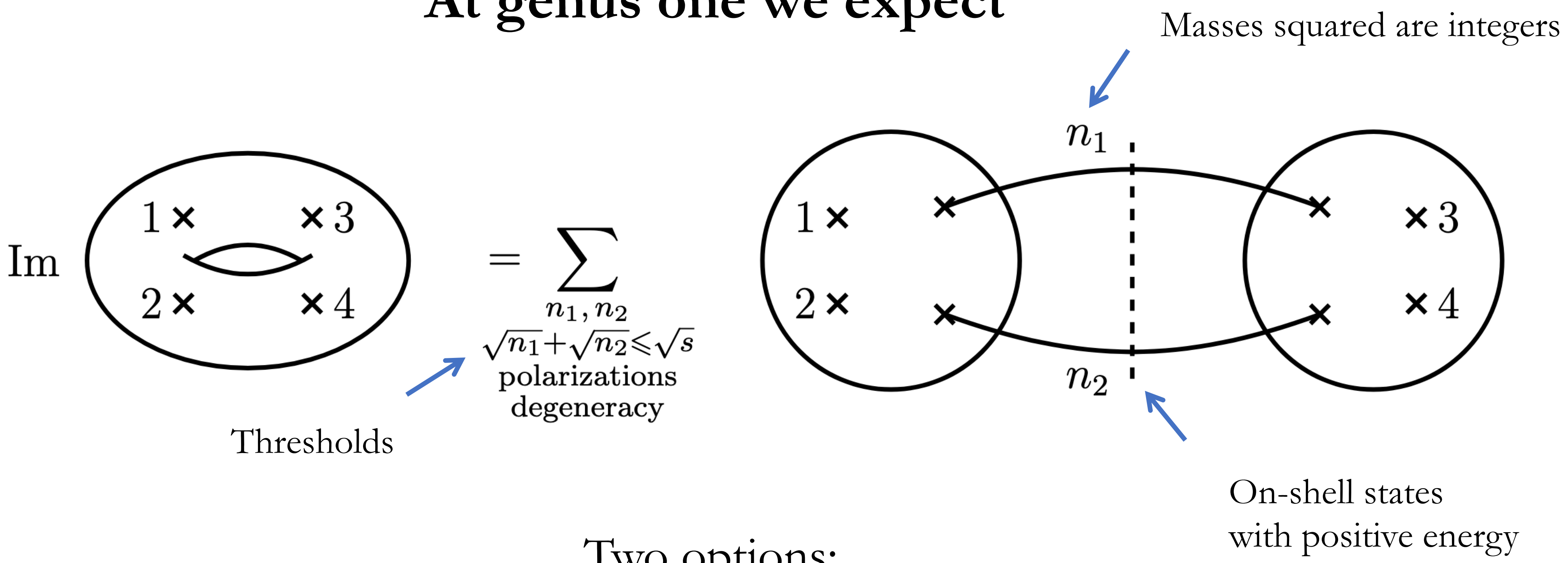


At genus one we expect



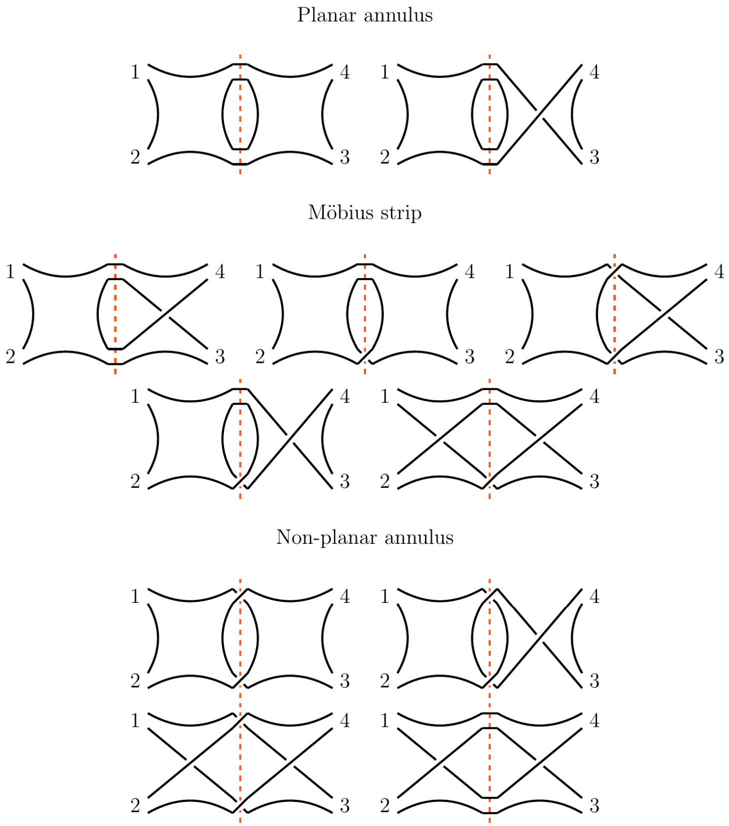
Two options:

- Do unitarity cuts "by hand" just as in field theory
- Let the worldsheet do it for us

First do it by hand

(not feasible beyond the massless cut)

- Color sums



- Polarization sums

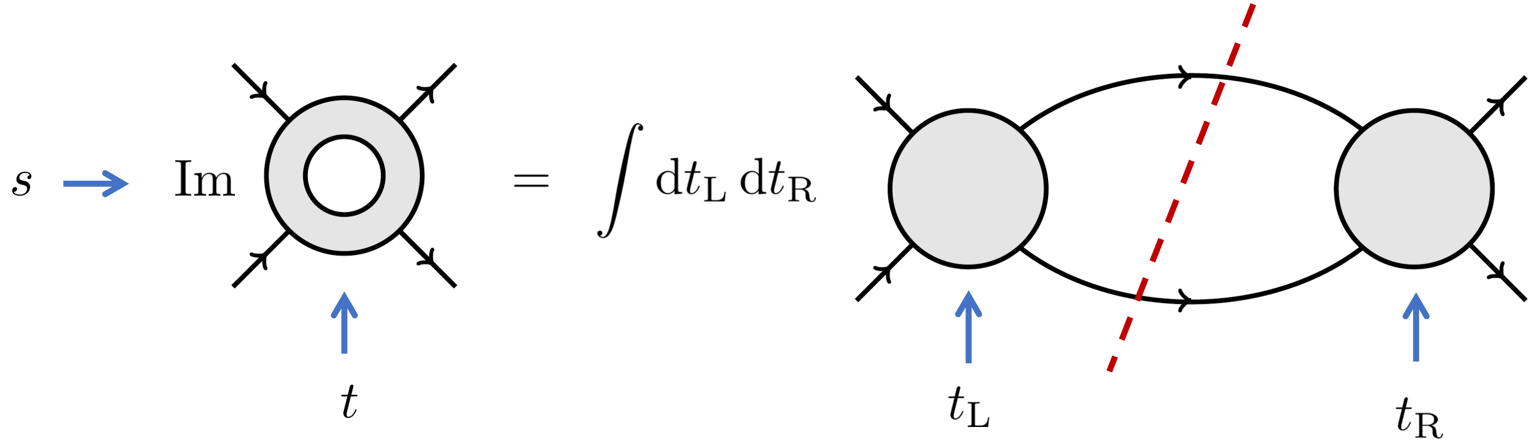
$$\mathcal{P} = \sum_{\text{pol}} \Big[\overbrace{t_8^b(1256)\, t_8^b(34\overline{56})}^{\text{gluons}} - \overbrace{t_8^f(1256)\, t_8^f(34\overline{56})}^{\text{gluinos}} \Big] = \frac{s^2}{2} t_8$$

- Loop integration

$$\int \mathrm{d}^{\mathrm{D}}\ell\, \delta^+[\ell^2]\delta^+[(p_{12}-\ell)^2](\cdots)$$

$$\propto \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}} P^{\frac{\mathrm{D}-5}{2}}(\cdots)$$

After the dust settles



$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}\Big|_{s<1} = \frac{N\pi}{60\sqrt{stu}} \int_{P>0} \mathrm{d}t_\mathrm{L}\, \mathrm{d}t_\mathrm{R}\;\; P(t_\mathrm{L}, t_\mathrm{R})^{\frac{5}{2}} \frac{\Gamma(1-s)\Gamma(-t_\mathrm{L})}{\Gamma(1-s-t_\mathrm{L})} \frac{\Gamma(1-s)\Gamma(-t_\mathrm{R})}{\Gamma(1-s-t_\mathrm{R})}$$

On-shell phase space

$$P(t_\mathrm{L}, t_\mathrm{R}) = -\frac{s(t^2 + t_\mathrm{L}^2 + t_\mathrm{R}^2 - 2tt_\mathrm{L} - 2tt_\mathrm{R} - 2t_\mathrm{L}t_\mathrm{R}) - 4tt_\mathrm{L}t_\mathrm{R}}{4tu}$$

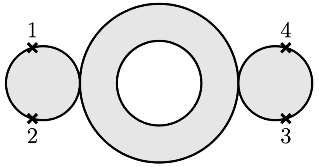
General form after including massive exchanges

New thresholds opening up

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi N}{60} \frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\big[s - (\sqrt{n_1}+\sqrt{n_2})^2\big] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}}$$

$$\times Q_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}}) \frac{\Gamma(-t_{\mathrm{L}})\Gamma(-t_{\mathrm{R}})}{\Gamma(n_1+n_2+1-s-t_{\mathrm{L}})\Gamma(n_1+n_2+1-s-t_{\mathrm{R}})}$$

Double poles at every positive integer



Need a computation to determine the integrand, e.g., $Q_{0,0} = 1$

with $P_{n_1,n_2} = -\frac{1}{4stu} \det \begin{bmatrix} 0 & s & u & n_2-s-t_{\mathrm{L}} \\ s & 0 & t & t_{\mathrm{L}}-n_1 \\ u & t & 0 & n_1-t_{\mathrm{R}} \\ n_2-s-t_{\mathrm{L}} & t_{\mathrm{L}}-n_1 & n_1-t_{\mathrm{R}} & 2n_1 \end{bmatrix}$

Shortcut computation using worldsheet methods

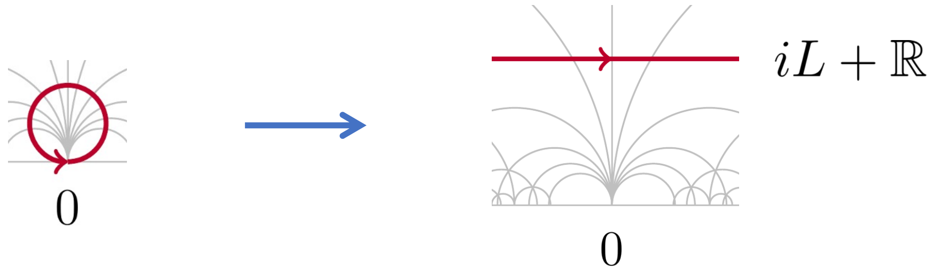
For the purposes of this talk, we only compute a toy model:

$$I = \int_{\circlearrowleft} \frac{\mathrm{d}\tau}{\eta(\tau)^{24}}$$

Dedekind eta function

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right)^{24} = (c\tau + d)^{12}\, \eta(\tau)^{24}$$

After modular transformation $\tau \to -1/\tau$:



$$I = -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{\tau^{14}\, \eta(\tau)^{24}}$$

Expand in large $\mathrm{Im}\,\tau$:

$$I = -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{\tau^{14}\,\eta(\tau)^{24}} = -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{\tau^{14}} \left(\mathrm{e}^{-2\pi i\tau} + \underbrace{24 + \mathcal{O}(\mathrm{e}^{2\pi i\tau})}_{\text{Exponentially suppressed}}\right)$$

Close the contour downstairs picking up the residue at $\tau = 0$:

$$I = -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{\tau^{14}} \mathrm{e}^{-2\pi i\tau} = \frac{(2\pi)^{14}}{13!}$$

Full worldsheet computation also has z_i moduli and evaluates to

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi N}{60}\frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\big[s - (\sqrt{n_1}+\sqrt{n_2})^2\big] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}}$$
$$\times Q_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}}) \frac{\Gamma(-t_{\mathrm{L}})\Gamma(-t_{\mathrm{R}})}{\Gamma(n_1+n_2+1-s-t_{\mathrm{L}})\Gamma(n_1+n_2+1-s-t_{\mathrm{R}})}$$

which gives us the polynomials we needed, e.g.,

$$\begin{aligned}
Q_{0,0} &= 1\ ,\\
Q_{1,0} &= 2\left(-2st_{\mathrm{L}}t_{\mathrm{R}} - s^2t_{\mathrm{L}} + st_{\mathrm{L}} - s^2t_{\mathrm{R}} + st_{\mathrm{R}} + s^2t - 2st + t\right)\ ,\\
Q_{2,0} &= 2s^4t_{\mathrm{L}}t_{\mathrm{R}} + 4s^3t_{\mathrm{L}}t_{\mathrm{R}}^2 + 4s^3t_{\mathrm{L}}^2t_{\mathrm{R}} - 4s^3tt_{\mathrm{L}}t_{\mathrm{R}} - 12s^3t_{\mathrm{L}}t_{\mathrm{R}} + 4s^2t_{\mathrm{L}}^2t_{\mathrm{R}}^2 - 10s^2t_{\mathrm{L}}t_{\mathrm{R}}^2\\
&\quad - 10s^2t_{\mathrm{L}}^2t_{\mathrm{R}} + 12s^2tt_{\mathrm{L}}t_{\mathrm{R}} + 18s^2t_{\mathrm{L}}t_{\mathrm{R}} - 2st_{\mathrm{L}}^2t_{\mathrm{R}}^2 + 4st_{\mathrm{L}}t_{\mathrm{R}}^2 + 4st_{\mathrm{L}}^2t_{\mathrm{R}} - 12stt_{\mathrm{L}}t_{\mathrm{R}}\\
&\quad - 6st_{\mathrm{L}}t_{\mathrm{R}} + 4tt_{\mathrm{L}}t_{\mathrm{R}} + s^4t_{\mathrm{L}}^2 - 2s^4tt_{\mathrm{L}} - s^4t_{\mathrm{L}} - 4s^3t_{\mathrm{L}}^2 + 10s^3tt_{\mathrm{L}} + 4s^3t_{\mathrm{L}} + 5s^2t_{\mathrm{L}}^2\\
&\quad - 18s^2tt_{\mathrm{L}} - 5s^2t_{\mathrm{L}} - 2st_{\mathrm{L}}^2 + 14stt_{\mathrm{L}} + 2st_{\mathrm{L}} - 4tt_{\mathrm{L}} + s^4t_{\mathrm{R}}^2 - 2s^4tt_{\mathrm{R}} - s^4t_{\mathrm{R}}\\
&\quad - 4s^3t_{\mathrm{R}}^2 + 10s^3tt_{\mathrm{R}} + 4s^3t_{\mathrm{R}} + 5s^2t_{\mathrm{R}}^2 - 18s^2tt_{\mathrm{R}} - 5s^2t_{\mathrm{R}} - 2st_{\mathrm{R}}^2 + 14stt_{\mathrm{R}}\\
&\quad + 2st_{\mathrm{R}} - 4tt_{\mathrm{R}} + s^4t^2 + s^4t - 6s^3t^2 - 6s^3t + 13s^2t^2 + 13s^2t - 12st^2 - 12st\\
&\quad + 4t^2 + 4t\ .
\end{aligned}$$

Encode the spectrum of type-I superstring (computed up to $s \lesssim 40$)

Bottom line:

Everything converges and can be computed with arbitrary precision

(plots later on)

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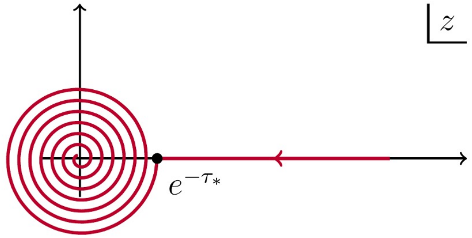

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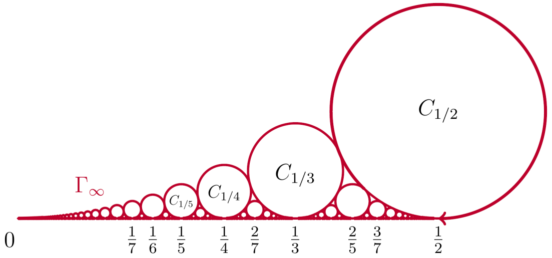

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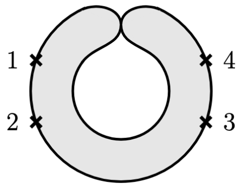

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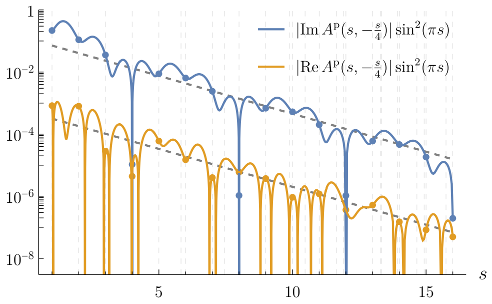
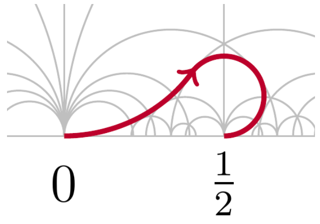
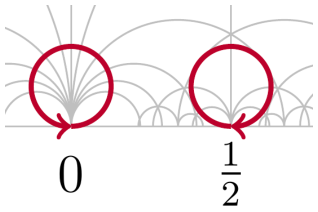


Recap:

The full genus-one amplitude is computed by



We just computed its imaginary part with



The idea is to recycle this computation (infinitely) many times

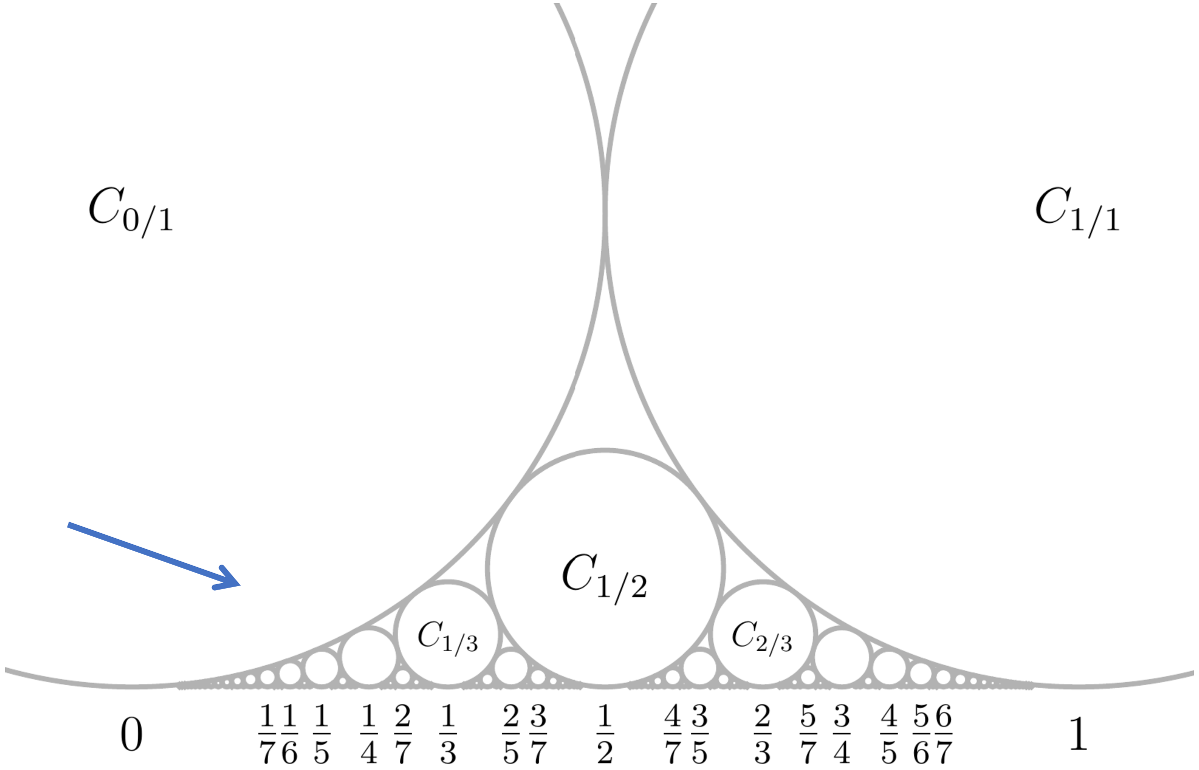
Farey sequence

$F_q =$ all irreducible fractions between 0 and 1 with the denominator $\leqslant q$

$$
\begin{aligned}
F_1 &= \left(\tfrac{0}{1}, \tfrac{1}{1}\right) \\
F_2 &= \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) \\
F_3 &= \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) \\
F_4 &= \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) \\
F_5 &= \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right) \\
&\vdots
\end{aligned}
$$

Ford circles

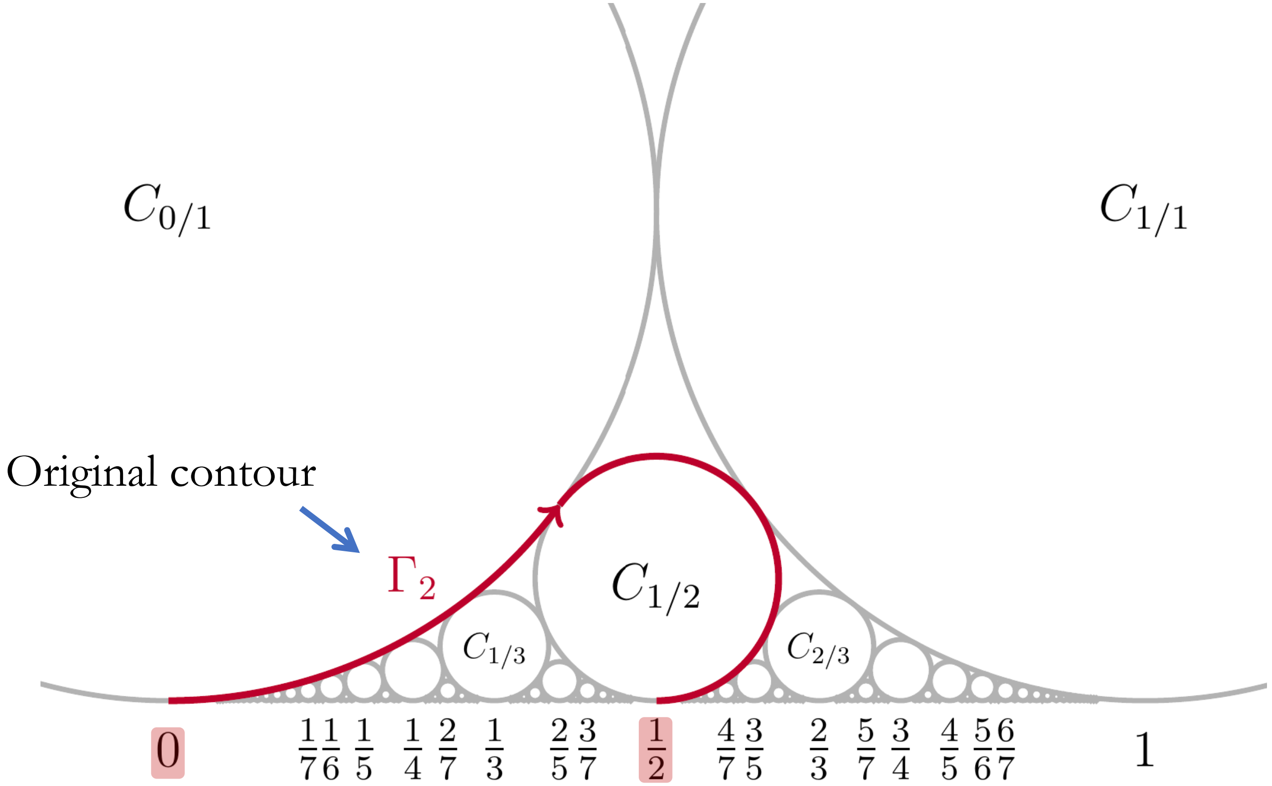
$C_{p/q} =$ circle touching the real axis at $\frac{p}{q}$ with radius $\frac{1}{2q^2}$ in the τ plane



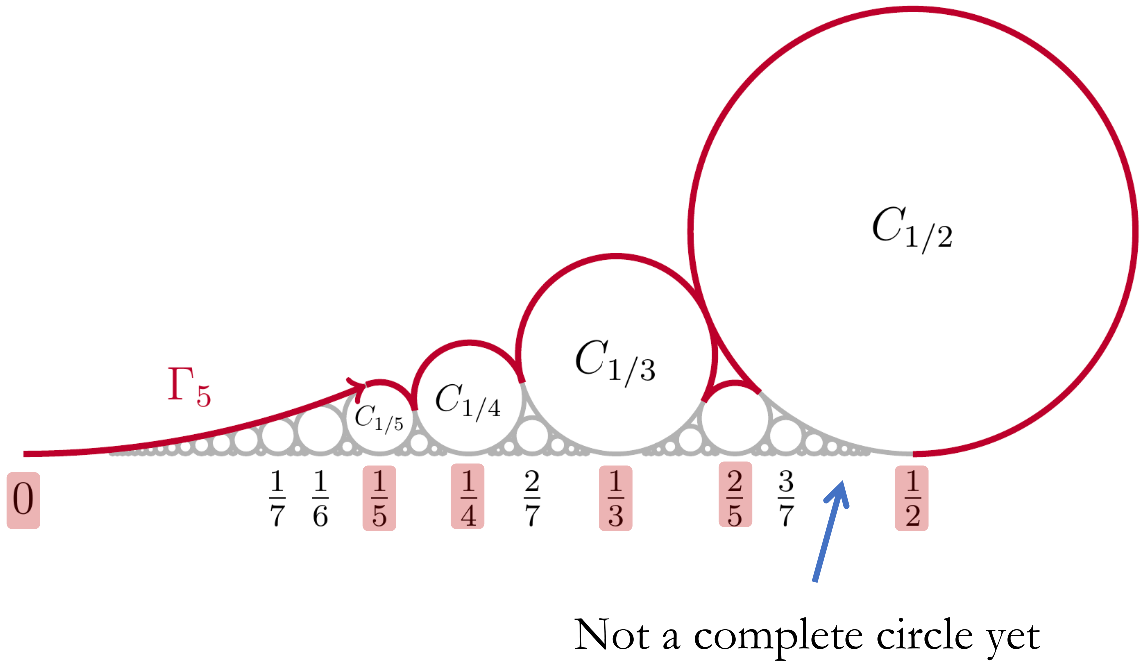
Each one is a modular transform of $C_{0/1}$

Rademacher contour

$\Gamma_q =$ follow all the Ford circles in the Farey sequence F_q from 0 to ½

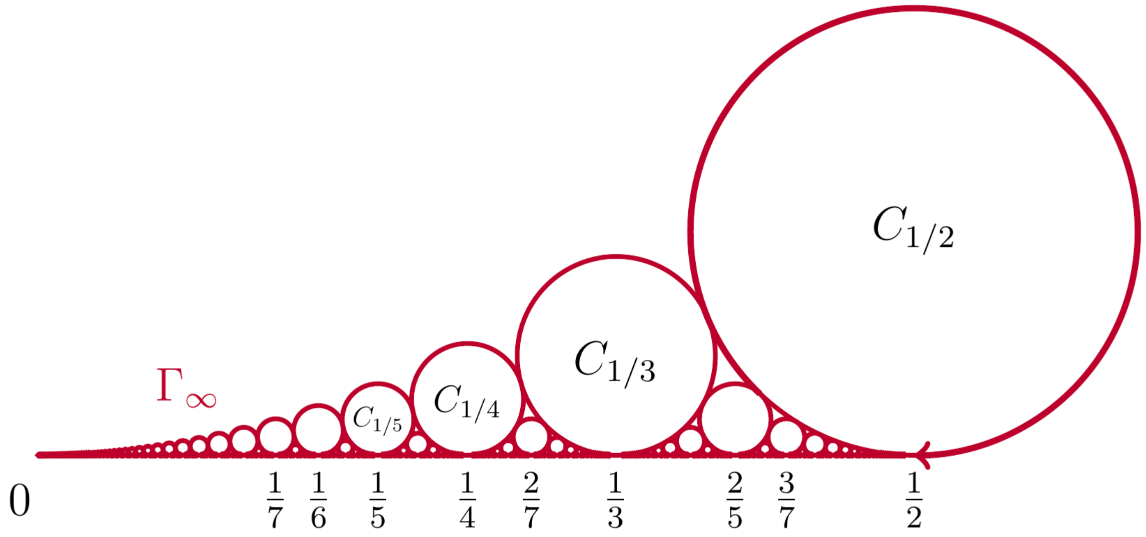


... and so on



Γ_5
$C_{1/5}$
$C_{1/4}$
$C_{1/3}$
$C_{1/2}$
0
$\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{2}{7}$ $\frac{1}{3}$ $\frac{2}{5}$ $\frac{3}{7}$ $\frac{1}{2}$
Not a complete circle yet

In the limit, we enclose all the circles



We call it the Rademacher contour

[Rademacher '43]

Back to the toy model

(bosonic open string partition function)

$$Z = -i \int_{\Gamma_\infty} \frac{\mathrm{d}\tau}{\eta(\tau)^{24}} = -i \sum_{c=1}^{\infty} \sum_{\substack{1 \leqslant a \leqslant \frac{c}{2} \\ (a,c)=1}} \underbrace{\int_{C_{a/c}} \frac{\mathrm{d}\tau}{\eta(\tau)^{24}}}$$

Recycle previous manipulations

$$\int_{C_{a/c}} \frac{\mathrm{d}\tau}{\eta(\tau)^{24}} = -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{(c\tau+d)^{14}\,\eta(\tau)^{24}}$$

Modular transformation ($ad \equiv 1 \mod c$)

$$= -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{(c\tau+d)^{14}} \left(\mathrm{e}^{-2\pi i\tau} + 24 + \mathcal{O}(\mathrm{e}^{2\pi i\tau})\right)$$

Push the horizontal contour all the way up

$$= -\int_{\longrightarrow} \frac{\mathrm{d}\tau}{(c\tau+d)^{14}} \mathrm{e}^{-2\pi i\tau}$$

Only one term survives

$$= 2\pi i \operatorname*{Res}_{\tau=-\frac{d}{c}} \frac{\mathrm{e}^{-2\pi i\tau}}{(c\tau+d)^{14}} = \frac{(2\pi)^{14}\mathrm{e}^{\frac{2\pi i d}{c}}}{13!\,c^{14}} \,.$$

Evaluate by residues

The result is a fast-convergent series expansion

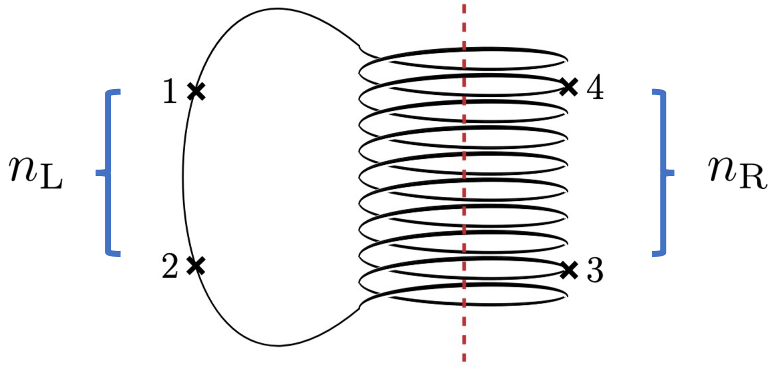
$$Z = \frac{-i(2\pi)^{14}}{13!} \sum_{c=1}^{\infty} \frac{1}{c^{14}} \sum_{\substack{1\leqslant a\leqslant \frac{c}{2} \\ (a,c)=1}} \mathrm{e}^{\frac{2\pi i a^*}{c}}$$

The full computation is technically much more involved, but conceptually similar; the final result is

$$A^{\mathrm{p}} = \Delta A^{\mathrm{p}} + \sum_{c=1}^{\infty} \sum_{\substack{1 \leqslant a \leqslant \frac{c}{2} \\ (a,c)=1}} \sum_{\substack{n_{\mathrm{L}}, n_{\mathrm{D}}, n_{\mathrm{R}}, n_{\mathrm{U}} \geqslant 0 \\ n_{\mathrm{L}}+n_{\mathrm{D}}+n_{\mathrm{R}}+n_{\mathrm{U}}=c-1}} A_{a/c}^{n_{\mathrm{L}}, n_{\mathrm{D}}, n_{\mathrm{R}}, n_{\mathrm{U}}}$$

Cusp contribution (easy)

Every term can be interpreted as summing over c windings with punctures distributed on the folds:



$$(n_{\mathrm{L}}, n_{\mathrm{D}}, n_{\mathrm{R}}, n_{\mathrm{U}}) = (0, 1, 7, 1)$$

Each term is almost the same as before

Sawtooth function $((x)) = x - \lfloor x \rfloor - \frac{1}{2}$
(arises because of infinite number of branch cuts)

$$A^{n_{\mathrm{L}},n_{\mathrm{D}},n_{\mathrm{R}},n_{\mathrm{U}}}_{a/c} = -\frac{16\pi i\, \mathrm{e}^{-\pi i \sum_{a=\mathrm{L,R},\, b=\mathrm{D,U}} \left[s \sum_{m=n_a+1}^{n_a+n_b} + t \sum_{m=n_b+1}^{n_a+n_b}\right] ((\frac{md}{c}))}}{15c^5 \sqrt{stu}} \sum_{\substack{m_{\mathrm{D}}, m_{\mathrm{U}} \geqslant 0 \\ (\sqrt{m_{\mathrm{D}}}+\sqrt{m_{\mathrm{U}}})^2 \leqslant s}}$$

Integrate over the phase space (manifestly convergent)

$$\times\, \mathrm{e}^{\frac{2\pi i d}{c}(m_{\mathrm{D}} n_{\mathrm{D}} + m_{\mathrm{U}} n_{\mathrm{U}})} \int_{P_{m_{\mathrm{D}},m_{\mathrm{U}}}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\; P_{m_{\mathrm{D}},m_{\mathrm{U}}}(s,t,t_{\mathrm{L}},t_{\mathrm{R}})^{\frac{5}{2}}\, Q_{m_{\mathrm{D}},m_{\mathrm{U}}}(s,t,t_{\mathrm{L}},t_{\mathrm{R}})$$

$$\times \left(\frac{\Gamma(-t_{\mathrm{L}})\Gamma(s+t_{\mathrm{L}}-m_{\mathrm{D}}-m_{\mathrm{U}})}{\Gamma(s)} \begin{cases} \mathrm{e}^{2\pi i t_{\mathrm{L}} ((\frac{d n_{\mathrm{L}}}{c}))} & \text{if } n_{\mathrm{L}} > 0 \\ \frac{\sin(\pi(s+t_{\mathrm{L}}))}{\sin(\pi s)} & \text{if } n_{\mathrm{L}} = 0 \end{cases} \right) (\mathrm{L} \leftrightarrow \mathrm{R})$$

Glue two Veneziano amplitudes with extra phases

Convergence

$$A^{\mathrm{p}} = \Delta A^{\mathrm{p}} + \sum_{c=1}^{\infty} \sum_{\substack{1 \leqslant a \leqslant \frac{c}{2} \\ (a,c)=1}} \sum_{\substack{n_{\mathrm{L}}, n_{\mathrm{D}}, n_{\mathrm{R}}, n_{\mathrm{U}} \geqslant 0 \\ n_{\mathrm{L}}+n_{\mathrm{D}}+n_{\mathrm{R}}+n_{\mathrm{U}}=c-1}} A_{a/c}^{n_{\mathrm{L}}, n_{\mathrm{D}}, n_{\mathrm{R}}, n_{\mathrm{U}}}$$

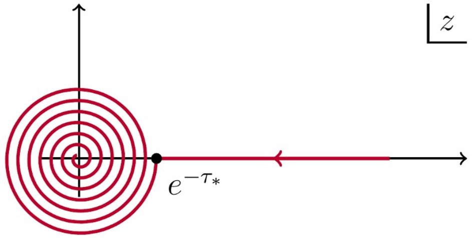
4 sums $\sim c^4$

$$\sim \frac{\mathrm{e}^{i(\text{phases})}}{c^5}$$

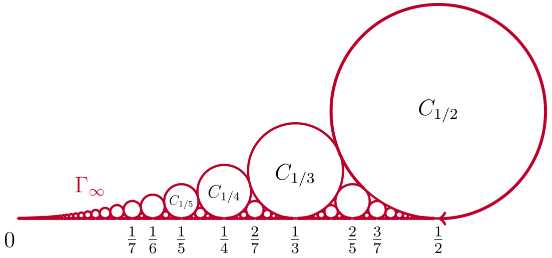
- Worst-case scenario (all phases vanish) : logarithmic divergence, $\alpha' \to 0$
- Best-case scenario (random phases): converges as $\sim \sum_{c=1}^{\infty} \frac{1}{c^3}$
- True rate of convergence somewhere in the middle

Outline of the talk

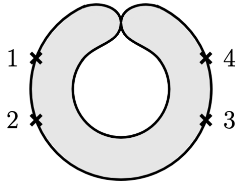
1) From Euclidean to Lorentzian worldsheets



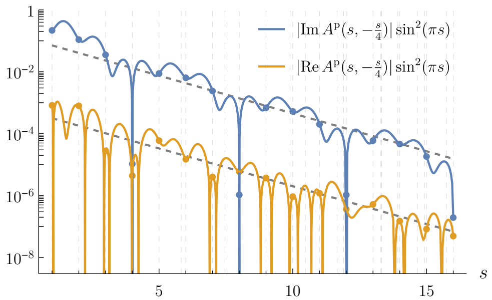
3) Rademacher expansion



2) Unitarity cuts of the worldsheet



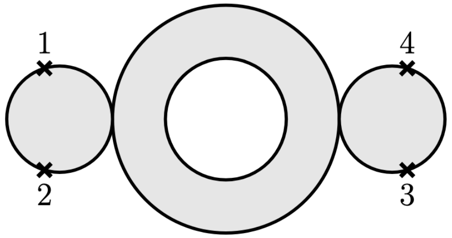
4) Physics of one-loop amplitudes



We can now analyze the results

(this talk: planar amplitudes in the s-channel only)

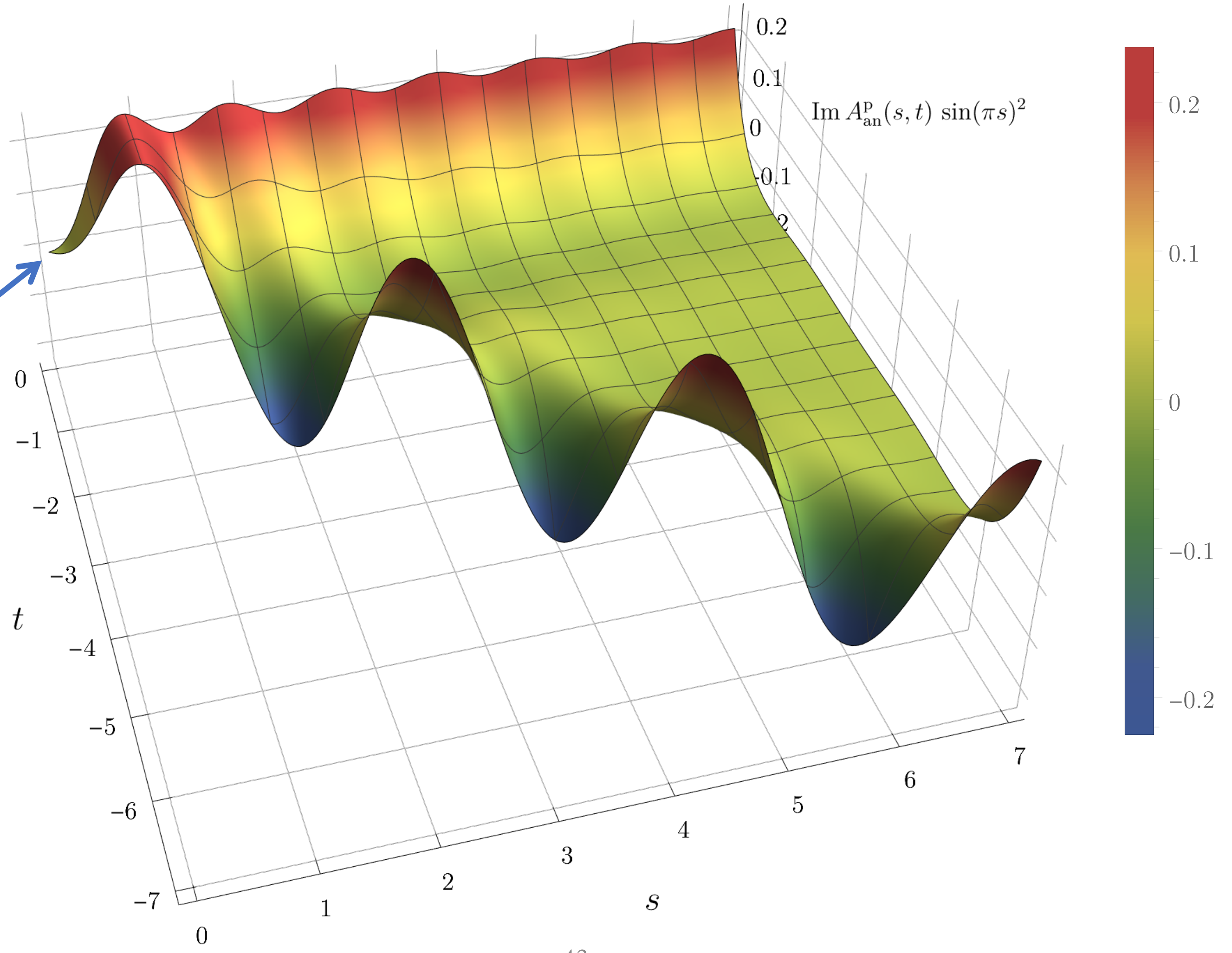
We often normalize by $\sin(\pi s)^2$ to remove the double poles



$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,t)$ does not include the t_8 tensor

First, just the imaginary part of the planar annulus

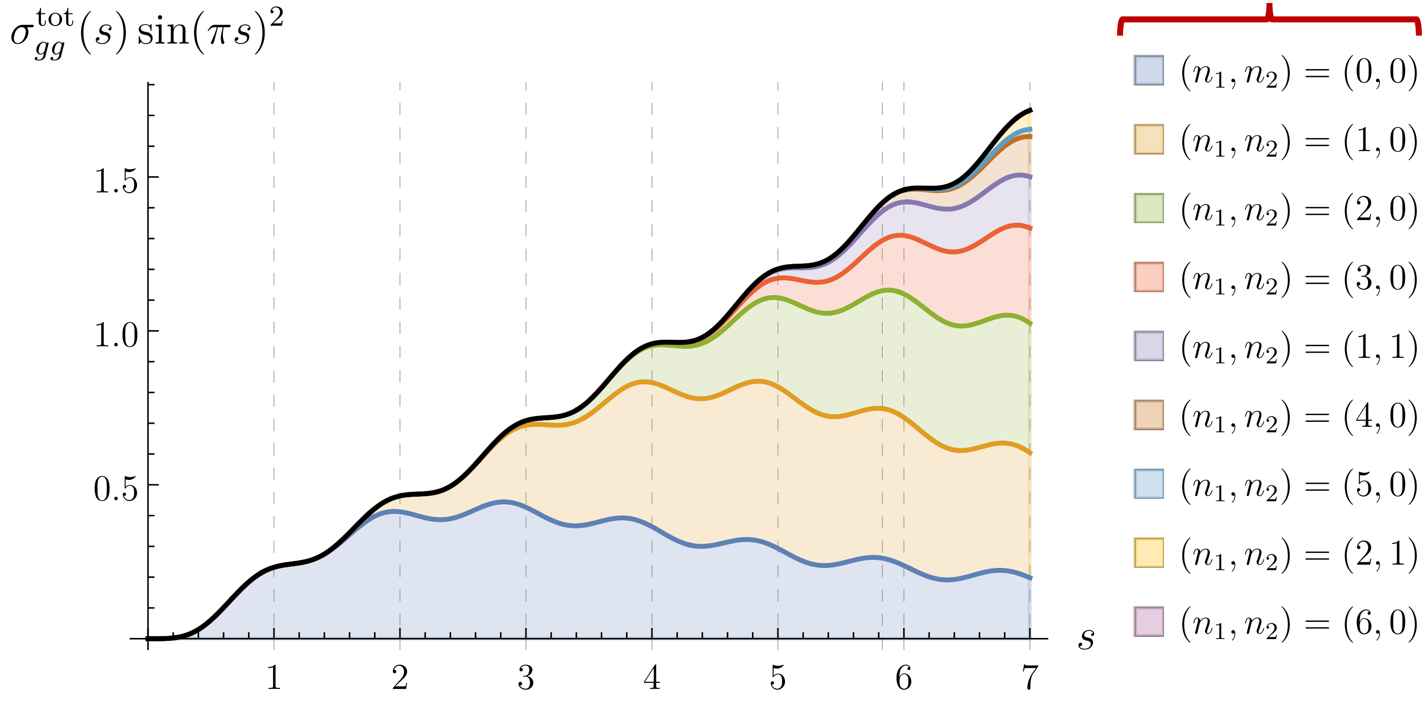
Most of the known results



Total cross section

Contribution from masses $\sqrt{n_1}$ and $\sqrt{n_2}$ flowing through the unitarity cut

$\propto \frac{t_8}{s} \mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s, 0)$

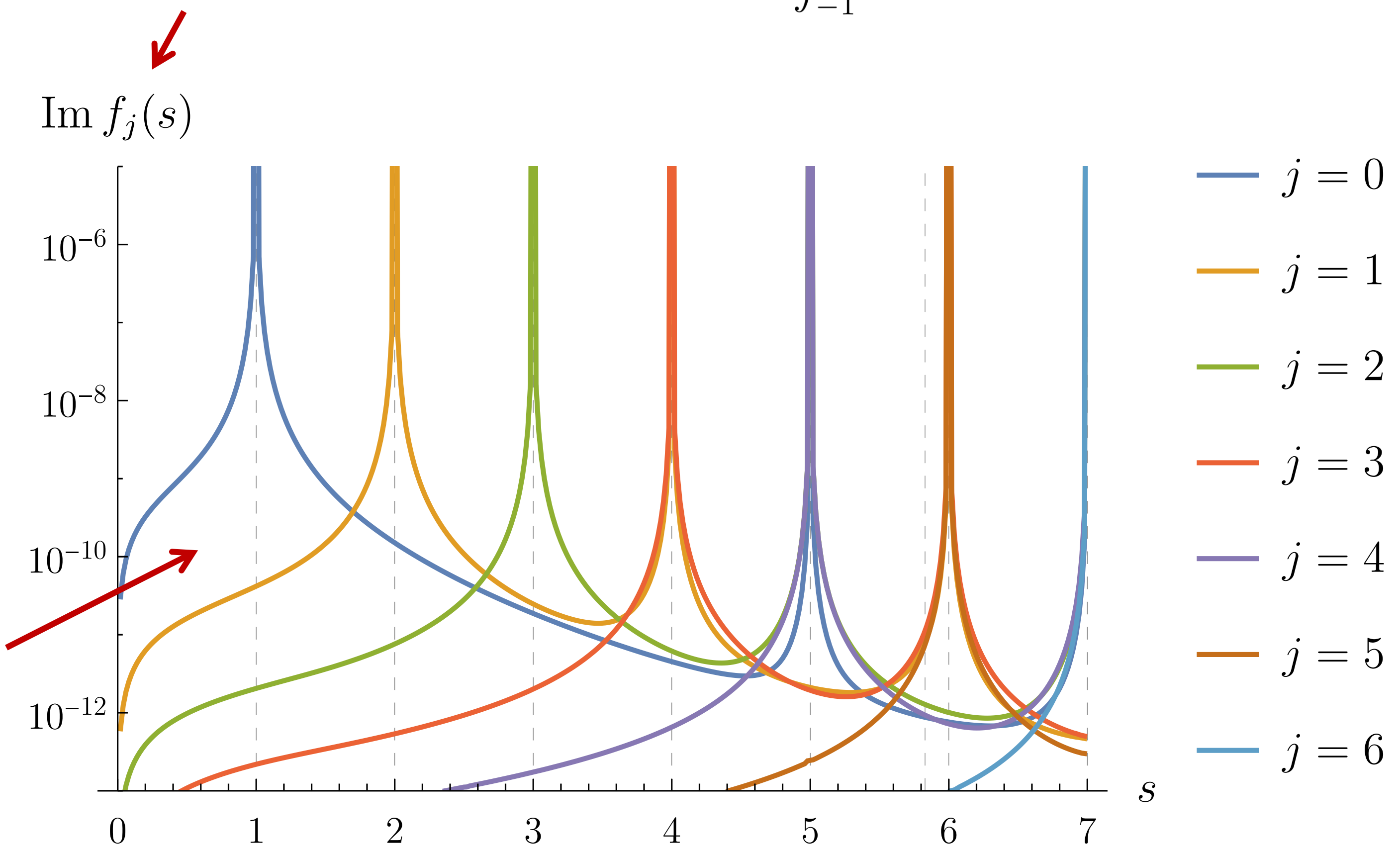


New thresholds opening up very slowly

Low-spin dominance

(cf. [Arkani-Hamed, Huang, Huang '20], [Bern, Kosmopoulos, Zhiboedov '21] at tree level)

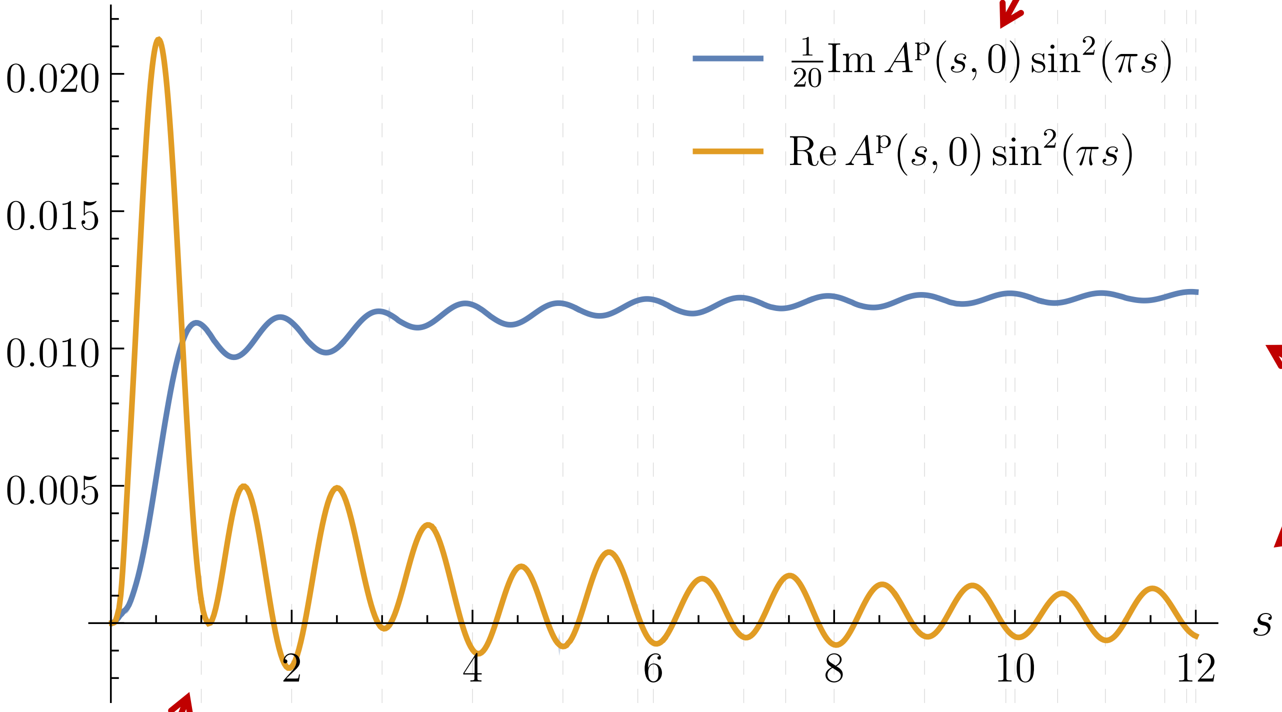
Partial wave coefficients $f_j(s) \propto \int_{-1}^{1} \mathrm{d}z\,(1-z^2)^3\, G_j^{(10)}(z) A^{\mathrm{p}}_{\mathrm{an}}(s,t(z))$



Almost all contributions from spins $j+1 \lesssim s$

Comparing the real and imaginary parts

Forward limit, $t = 0$

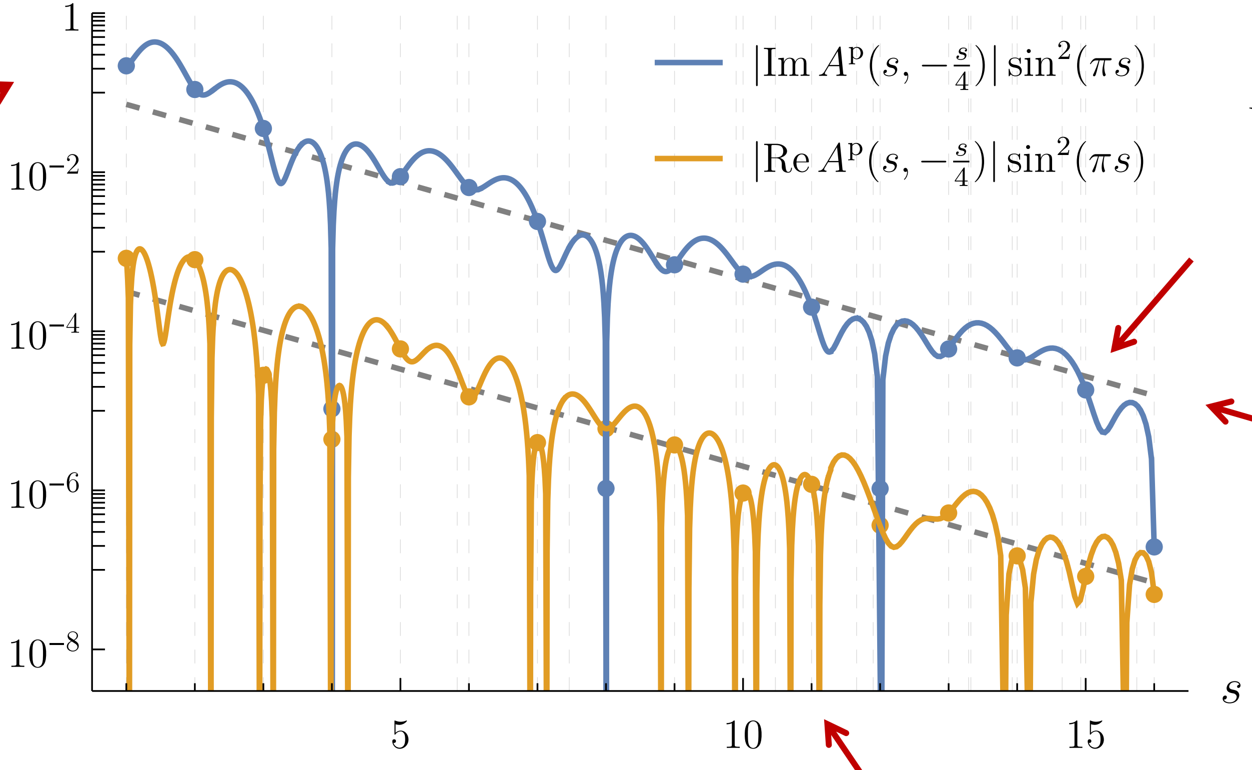


The real part suppressed by ~2 orders of magnitude

Error bars smaller than the line widths (truncate at $c \approx 40$ and extrapolate)

Fixed scattering angle, $\cos\theta = 1 + 2t/s$

Logarithmic scale



Values at integer s computed with arbitrary precision (related to decay widths and mass shifts)

Exponential suppression

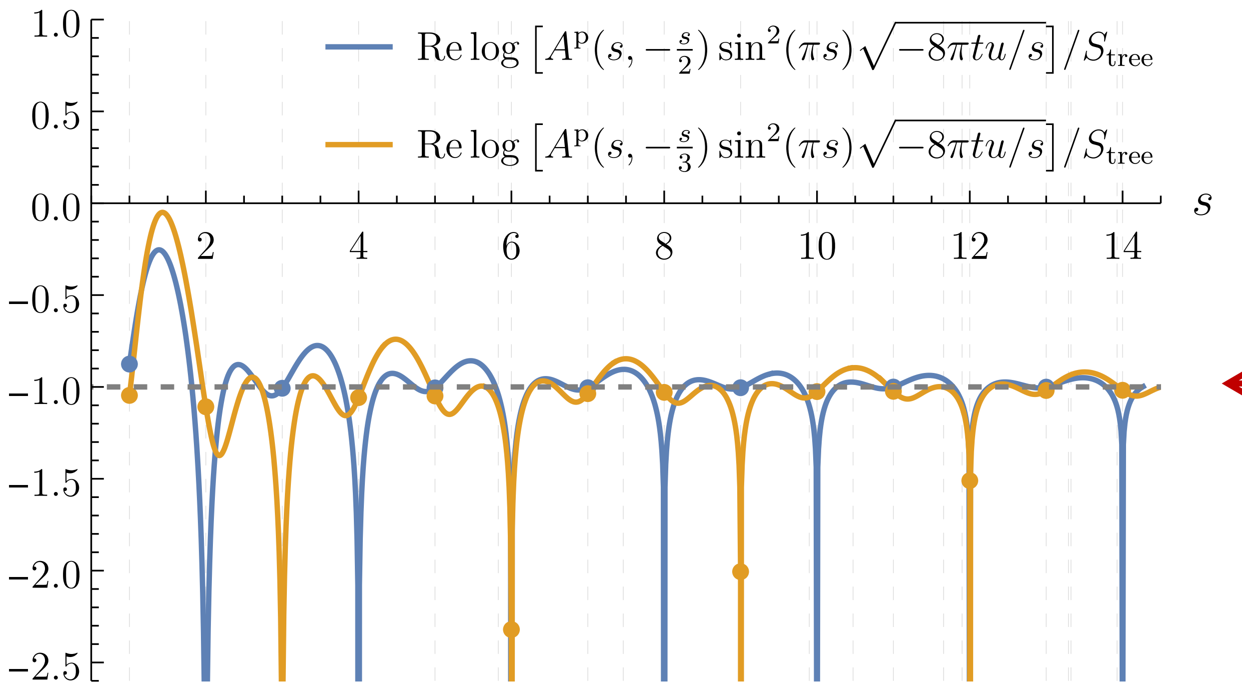
Zeros

Test an old-standing conjecture

Heuristic arguments: $A^{\mathrm{p}} \approx \mathrm{e}^{-\alpha' S_{\mathrm{tree}}}$ as $\alpha' \to \infty$, where

$S_{\mathrm{tree}} = s\log(s) + t\log(-t) + u\log(-u)$ is the tree-level on-shell action

[Gross, Mende, Manes '87-89]



Exponential suppression

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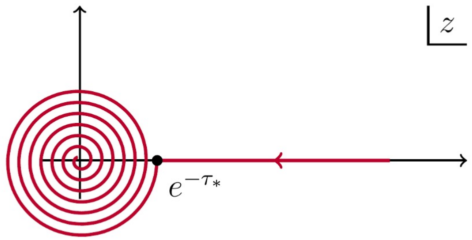

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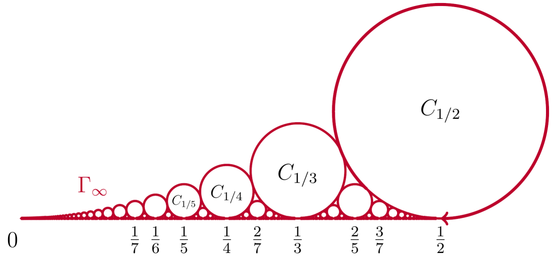

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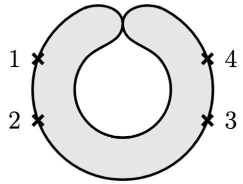

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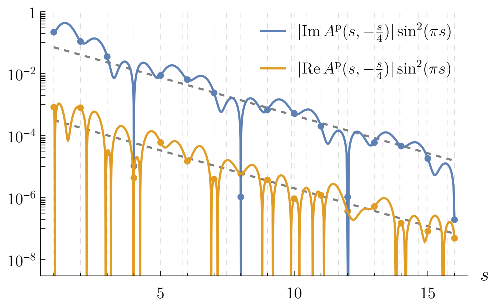

Thank you!