Bounds on Crossing Symmetry

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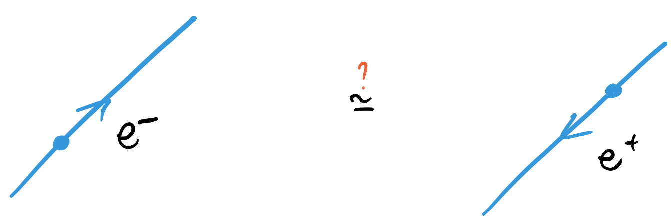
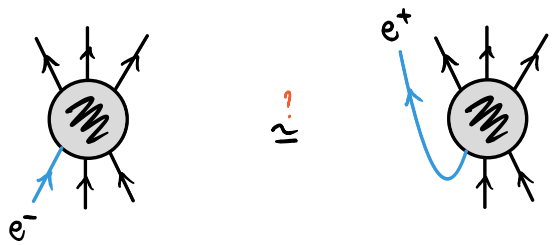
hep-th/2101.08266

What is crossing symmetry?



$p^\mu = (E, \vec{p})$ $\qquad\qquad$ $-p^\mu = (-E, -\vec{p})$

On the level of observables:



for any number and type of the remaining particles

Are scattering amplitudes in different crossing channels boundary values of the same function?

→ Proposed in 1954 by Gell-Mann, Goldberger, and Thirring

→ As of 2021 we still don't know if it's true or not

If you're interested in the history of science:

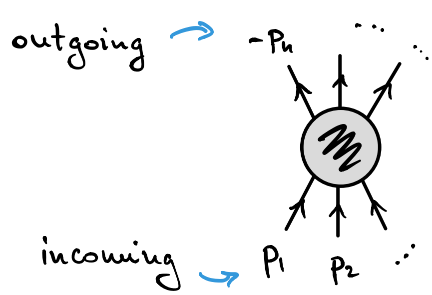
Murray Gell-Mann

Web of Stories

As a matter of introduction, let's ask:

- What is known about crossing symmetry, really?
- Why is it difficult to prove it in general?
- What can we do about it?

Conventions:



- momentum conservation reads $\sum_{i=1}^{n} p_i^\mu = 0$ and masses $p_i^2 = M_i^2$.
- $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.

It's believed that the following physical assumptions will be needed in the proof:

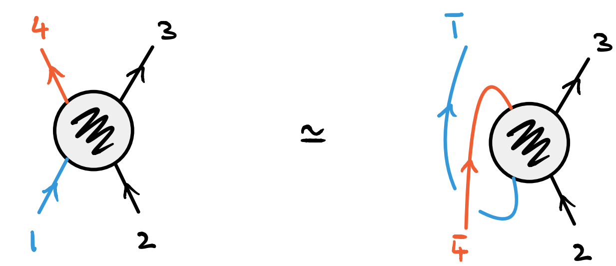
→ Locality (Existence of local operators $O(x)$)
→ Causality ($[O(x), O(y)] = 0$ when $(x-y)^2 < 0$)
→ Unitarity ($\mathbb{1} = \int d^4\tilde{p}\ |p\rangle\langle p|$)

Still not enough! We also need:

→ Mass gap (no massless particles)

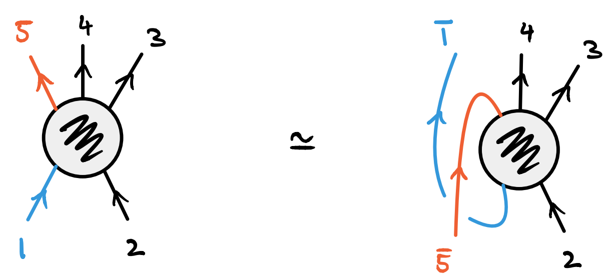
With these assumptions one can show crossing (non-perturbatively) between:

$2 \to 2$:



[Bros, Epstein, Glaser '65]

$2 \to 3$:



[Bros '86]

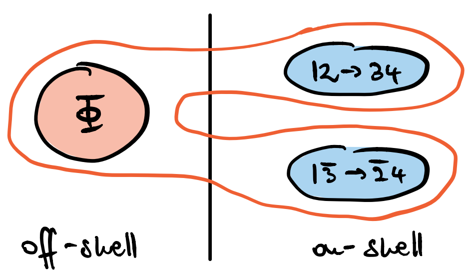
The proof proceeds in 3 steps

[Steinmann, Ruelle, Araki, Burgoyne,
Bros, Epstein, Glaser '60-86] ~ 200 pages

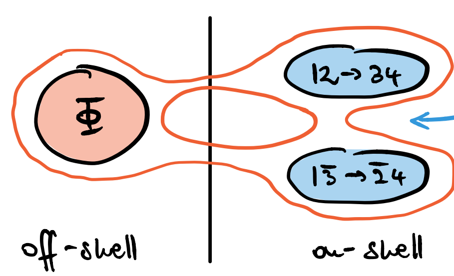
1. easy Locality, causality, unitarity, mass gap
+ LSZ procedure $p_i^2 < 0$
=> analyticity in an off-shell region Φ

at this stage we can forget about physics!

2. easy Envelope of holomorphy of Φ (any multiplicity)



3. hard For 2→2 and 2→3 processes: Envelope of holomorphy of Φ has an on-shell continuation



$|s_{COM}| \to \infty$

crossing symmetry!

Why is this not satisfactory?

→ Doesn't give us any physics understanding:
the proof is mostly complex analysis

→ Prohibitively technical:
poor prospects for a general proof

The simplest cases where crossing symmetry has never been proven:

→ Any process involving massless particles

→ Different number of in/out states, e.g.,

$$12 \to 34 \quad \text{and} \quad 1 \to \bar{2}34$$

$$12 \to 345 \quad \text{and} \quad 12\bar{3} \to 45$$

→ Any amplitude with $n > 5$ external particles

As an aside, the standard approach since the 1960s has been to ignore this issue and replace it with a much stronger assumption:

- Locality
- Causality
- Unitarity
- Analyticity

$\Rightarrow$ (trivially) crossing symmetry

mathematical assumption

In my opinion, mathematical assumptions shouldn't form a basis for a physical theory.
We should scrutinize the extent to which analyticity follows from physical axioms.

What can we do about this situation?

The idea (due to Witten) is to prove crossing symmetry on-shell within the framework of perturbation theory, where one might reasonably hope to circumvent the aforementioned issues.

→ No known counterexamples
→ Standard ways of dealing with UV/IR divergences
→ Same Feynman rules for any n
→ Analyticity governed by Landau equations

This is clearly an ambitious goal. The purpose of this talk is to convince you that it isn't completely outlandish.

We'll show that crossing symmetry is true to all loops and multiplicities provided that the internal states are not too light:

$$m_e > \frac{\sqrt{n}}{2\sqrt{2}} M_{bound}$$

internal masses

$\mathcal{O}(1)$ number

where

$$M_{bound} = \sqrt{\max_i \left(M_i^2 , \frac{\Sigma_j M_j^2 - 2M_i^2}{n-2} \right)}$$

external masses

Comments:

→ Includes cases not known non-perturbatively: different number of in/out states, massless particles, cases $n > 5$

→ When all external particles are massless, crossing is true for arbitrary non-zero internal masses

→ Scattering of the lightest state up to $n < 8$.

→ The case $n=4$ was known since the early years of dispersion relations

→ The cases $n \geq 6$ require analytic continuation into higher dimensions

→ In perturbation theory, it's currently unknown if crossing symmetry holds beyond these bounds

For a CPT-symmetric theory, Feynman diagrams have a crossing property

$$\int d^{4L} \ell_i \sum_{F.D.} \frac{\text{numerators}}{\prod_e (q_e^2 - m_e^2 + i\epsilon)}$$

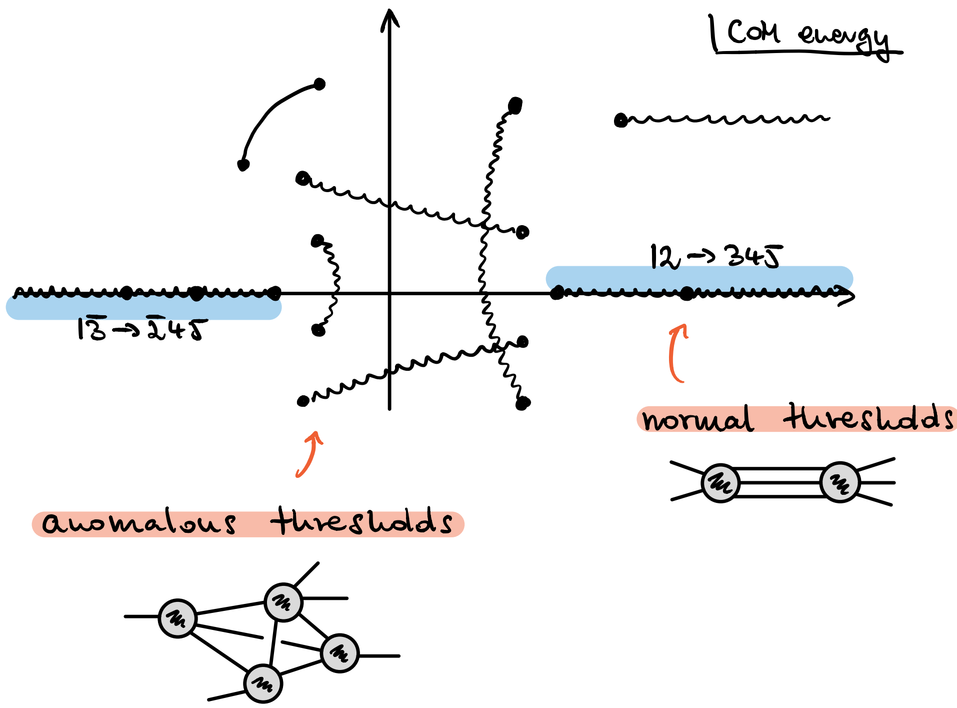
The trouble lies in demonstrating that Feynman integrals are finite everywhere along a path of continuation between any pair of crossing channels.

As usual, we distinguish between

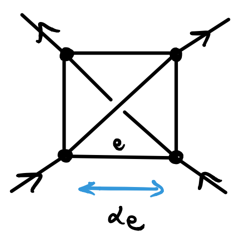
→ Overall divergences (UV/IR) happen for all values of $p_i \cdot p_j$. Can be treated using standard tools such as BPHZ renormalization or dimensional regularization.

→ Singularities (particles going on-shell) happen for specific values of $p_i \cdot p_j$.
Spin effects cannot introduce new singularities, so it suffices to study scalar diagrams.

Crossing symmetry is related to a conjecture that there exist Riemann surfaces of complexified kinematic invariants, e.g.

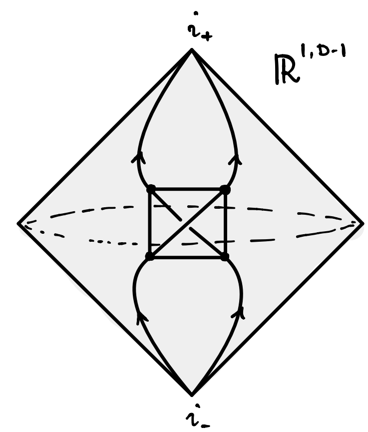


For the purpose of proving crossing symmetry it's the most convenient to work in the worldline formalism:



Schwinger parameters
= proper times

free scalar field X^μ



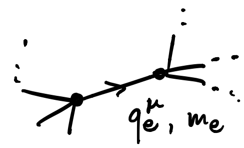
Feynman integral with E internal edges and L loops evaluates to

$$I = \int_0^\infty \frac{d^E \alpha_e}{\mathcal{U}^{D/2}} \; e^{\frac{i}{\hbar} V}$$

action V

determinant of Laplacian $\mathcal{U}$

As a reminder, you can also obtain it from the loop momentum picture:



$$I = \# \int d^{4L} \ell_i \prod_{e=1}^{E} \frac{i\hbar}{q_e^2 - m_e^2 + i\epsilon}$$

Schwinger trick

$$= \# \int d^{4L} \ell_i \, d^E \alpha_e \; e^{\frac{i}{\hbar} \sum_e (q_e^2 - m_e^2 + i\epsilon)\alpha_e}$$

Gaussian in ℓ_i^μ

$$= \# \int_0^\infty \frac{d^E \alpha_e}{\mathcal{U}^{D/2}} \; e^{\frac{i}{\hbar} V} .$$

We can also use the rescaling freedom $\alpha_e \to \lambda \alpha_e$ and integrate out the overall scale λ:

for theories with spin

$$I = \# \int_0^1 \frac{d^E \alpha_e \; \delta\left(\sum_e \alpha_e - 1\right)}{\mathcal{U}^{D/2} \; V^{E - LD/2}} \quad (\times \text{ polynomial})$$

we need to understand what $\mathcal{U}$ and V are

But before that let us discuss singularities, which happen in the classical limit, $\hbar \to 0$.

Saddle-point equations:

$$\underbrace{\alpha_e}\,\underbrace{\frac{\partial V}{\partial \alpha_e}} = 0 \qquad \text{for all } e.$$

putting the edge on-shell $q_e^2 = m_e^2$

contracting an edge

Since we already consider all possible diagrams, we can take $\alpha_e \neq 0$.

Because of homogeneity of the action $\left(V(\lambda\alpha_e) = \lambda V(\alpha_e)\right)$:

$$V = \sum_e \alpha_e \frac{\partial V}{\partial \alpha_e} = 0.$$

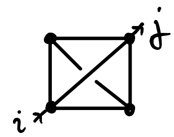
To summarize, singularities are determined by the (leading) Landau equations:

$$V = \frac{\partial V}{\partial \alpha_1} = \frac{\partial V}{\partial \alpha_2} = \dots = \frac{\partial V}{\partial \alpha_E} = 0.$$

Hence we're interested in the properties of the localized action V. It reads:

$$V = -\sum_{i<j} p_i \cdot p_j \, G_{ij} - \sum_e m_e^2 \alpha_e .$$

graph Green's function



$$\left(\begin{array}{l} G_{ii} = 0 \\ G_{ij} = G_{ji} \end{array} \right)$$

Now the problem looks like a superposition of many 2-pt problems subject to the momentum conservation $\sum_{i=1}^{n} p_i^\mu = 0$, and on-shell conditions $p_i^2 = M_i^2$.

(If it wasn't for the on-shell conditions, the problem would be trivial: deform in $\mathrm{Im}(p_i \cdot p_j) < 0$.)

$$\Updownarrow$$

$$M_i \to M_i + i\epsilon$$

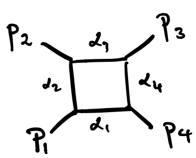
Explicit expressions:

$$\left(\begin{aligned} I &= \int \frac{d^E \alpha_e}{\mathcal{U}^{D/2}}\, e^{\frac{i}{\hbar} V} \\ V &= -\sum_{i<j} p_i \cdot p_j\, G_{ij} - \sum_e m_e^2 \alpha_e \end{aligned} \right)$$

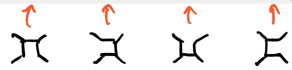
$$\mathcal{U} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \notin T} \alpha_e \,,$$

$$G_{ij} = \frac{1}{\mathcal{U}} \sum_{\substack{\text{spanning} \\ \text{2-forests } T_i \cup T_j}} \prod_{e \notin T_i, T_j} \alpha_e \,.$$

E.g. for the box diagram we have



$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \,,$$



$$G_{12} = \alpha_2(\alpha_1 + \alpha_3 + \alpha_4)\,,$$

$$G_{13} = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\,,$$

$$G_{14} = \alpha_1(\alpha_2 + \alpha_3 + \alpha_4)\,,$$

$$G_{23} = \alpha_3(\alpha_4 + \alpha_1 + \alpha_2)\,,$$

$$G_{24} = (\alpha_2 + \alpha_3)(\alpha_4 + \alpha_1)\,,$$

$$G_{34} = \alpha_4(\alpha_1 + \alpha_2 + \alpha_3)\,.$$

(Warning: planar diagrams have very simple analyticity prop^s. $O(n^2)$ vs. $O(2^n)$ channels)

Before discussing crossing symmetry, let us mention causality, which can be imposed with the $i\epsilon$ prescription:

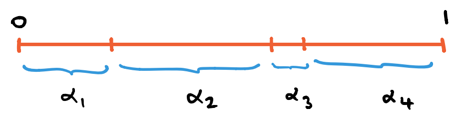
$$V \longrightarrow V + i\epsilon,$$

but we will instead deform the external kinematics to the same effect, $\mathrm{Im}\, V > 0$.

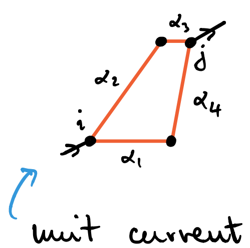
We'll need bound on the Green's functions G_{ij}.
Here's where the analogy with electric circuits (scalar field on a graph) comes useful:

[Bjorken, Mathews '59]

Take a unit-length wire with uniform resistivity



and build a Feynman-diagram circuit, e.g.



resistance $\sim$ α_e

current $\sim$ q_e

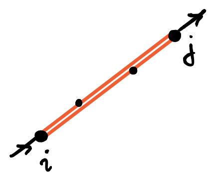
Kirchhoff's laws:

$$\sum_{e \in v} \pm q_e = \sum_{e \in \text{loop}} \pm \alpha_e q_e = 0.$$

$$G_{ij} = \sum_e \alpha_e q_e^2 = \text{power dissipated by the circuit}$$

Clearly $G_{ij} > 0$ and there must exist an upper bound. What is it?

Answer: align the wires in parallel:

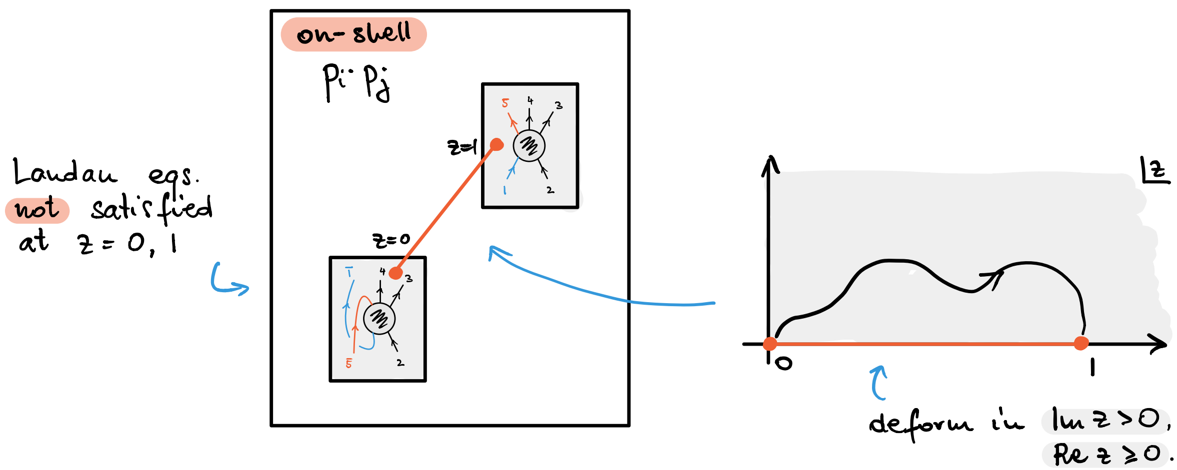


$$\text{Power} = \underbrace{2}_{\text{2 parallel wires}} \times \underbrace{\tfrac{1}{2}}_{\text{each length } 1/2} \times \underbrace{\left(\tfrac{1}{2}\right)^2}_{\text{current split evenly}} = \frac{1}{4}$$

Hence we have: $0 < G_{ij} \leq 1/4$.

independent of the number of loops!

Naive approach:



Interpolate linearly in the space of $p_i \cdot p_j$ invariants

$$p_i \cdot p_j = p_i^{(0)} \cdot p_j^{(0)} + z \left(p_i^{(1)} \cdot p_j^{(1)} - p_i^{(0)} \cdot p_j^{(0)} \right).$$

→ Preserves momentum conservation

→ Remains on-shell

→ Deforms into higher dimensions for $n \geq 6$ because

$$\mathrm{rank}\,(p_i \cdot p_j) = n-1 \leq D.$$

The action $V = -\sum_{i<j} p_i \cdot p_j \, G_{ij} - \sum_e m_e^2 \alpha_e$ responds linearly to changes in kinematics:

$$V = V_0 + z\,(V_1 - V_0).$$

evaluated @ $p_i \cdot p_j$ — @ $p_i^{(0)} \cdot p_j^{(0)}$ — @ $p_i^{(1)} \cdot p_j^{(1)}$

The real and imaginary parts of Landau equations:

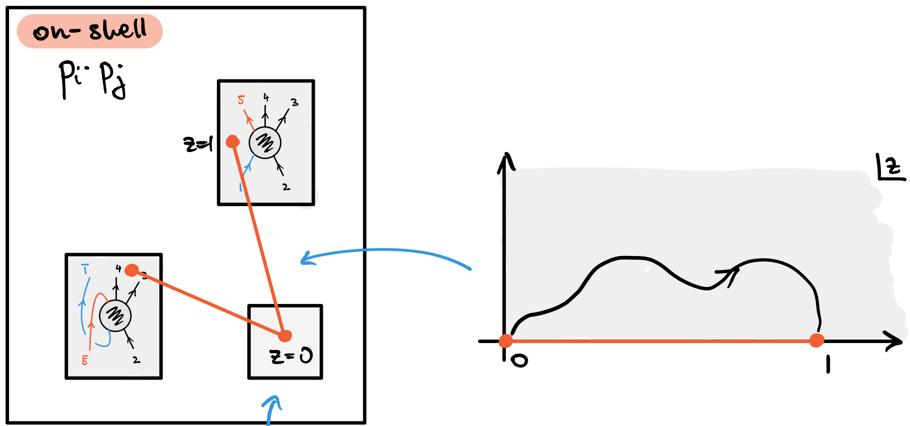
$$\underset{\neq\, 0}{\overset{\text{real}}{\frac{\partial V_0}{\partial \alpha_e}}} + \operatorname{Re} z \left(\overset{\text{real}}{\frac{\partial V_1}{\partial \alpha_e}} - \frac{\partial V_0}{\partial \alpha_e} \right) = 0\,,$$

$$\underset{>\, 0}{\operatorname{Im} z} \left(\frac{\partial V_1}{\partial \alpha_e} - \frac{\partial V_0}{\partial \alpha_e} \right) = 0 \qquad \text{for all } e.$$

These cannot be simultaneously satisfied for all e, because V_0 doesn't satisfy Landau equations by assumption.

However by deforming into complex plane we lost control over the sign of $\operatorname{Im} V$, i.e., causality isn't imposed ...

The fix involves continuing via an auxilliary region:



open set $\{p_i^{(*)} \cdot p_j^{(*)}\}$ where $V_* < 0$.

Deform $\quad p_i \cdot p_j = p_i^{(*)} \cdot p_j^{(*)} + z\left(p_i^{(1)} \cdot p_j^{(1)} - p_i^{(*)} \cdot p_j^{(*)} \right).$

so that $\quad V = V_* + z\,(V_1 - V_*).$

There are two options depending on the sign of $V_1 - V_*$:

$\rightarrow \quad V_1 > V_* \quad \Rightarrow \quad \mathrm{Im}\, V = \underbrace{\mathrm{Im}\, z}_{>0} \, \underbrace{(V_1 - V_*)}_{>0} > 0,$

(correct $i\epsilon$ prescription)

$\rightarrow \quad V_1 \leq V_* < 0 \quad \Rightarrow \quad \mathrm{Re}\, V = \underbrace{V_*}_{<0} + \underbrace{\mathrm{Re}\, z}_{\geq 0} \, \underbrace{(V_1 - V_*)}_{\leq 0} < 0.$

(no $i\epsilon$ prescription necessary)

It remains to show that a region $\{p_i^{(*)} \cdot p_j^{(*)}\}$ with $V_* < 0$ exists in the first place.

We'll take $-p_i^{(*)} \cdot p_j^{(*)} < c$ for some constant c and $i \neq j$.

Then:

$$V_* = \underbrace{\sum_{i<j}}_{\binom{n}{2} = \frac{n(n-1)}{2} \text{ terms}} \underbrace{\left(-p_i^{(*)} \cdot p_j^{(*)}\right)}_{<c} \underbrace{G_{ij}}_{\leq 1/4} - \sum_e m_e^2 \alpha_e$$

$$< \frac{n(n-1)}{8} c - \sum_e m_e^2 \alpha_e$$

Let's call the lightest internal mass $m = \min_e (m_e)$

$$V_* < \frac{n(n-1)}{8} c - m^2 \underbrace{\sum_e \alpha_e}_{=1}$$

Requiring $V_* < 0$ gives

$$-p_i^{(*)} \cdot p_j^{(*)} < c < \frac{8}{n(n-1)} m^2$$

Using on-shell conditions we have for every i:

$$M_i^2 = (p_i^{(*)})^2 = -\underbrace{p_i^{(*)} \cdot \sum_{j \neq i} p_j^{(*)}}_{n-1 \text{ terms}} < \frac{8}{n} m^2 .$$

and also

$$M_i^2 = \sum_{j \neq i} p_j^{(*)} \cdot \sum_{k \neq i} p_k^{(*)}$$

$$= \sum_{j \neq i} M_j^2 + \underbrace{\sum_{j \neq i} p_j^{(*)} \cdot \sum_{k \neq i,j} p_k^{(*)}}_{(n-1)(n-2) \text{ terms}}$$

This gives the other set of constraints

$$\sum_j M_j^2 - 2M_i^2 < \frac{8(n-2)}{n} m^2 .$$

To summarize, crossing symmetry holds to all loops in perturbation theory when

$$m_e > \frac{\sqrt{n}}{2\sqrt{2}} \sqrt{\max_i \left(M_i^2 , \frac{\Sigma_j M_j^2 - 2M_i^2}{n-2} \right)}$$

Summary

- We argued that a proof of crossing symmetry is an important open problem in QFT
- There's hope it can be solved in perturbation theory
- Our approach is based on extremizing the worldline action
- As a proof-of-principle we showed that crossing symmetry holds to all orders in perturbation theory provided that the internal states are not too light

Thank you!